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EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN
THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER
JOHN R. KLINE HENRY L. RIETZ

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SOME NEW COMPLETE SETS OF IDENTITIES FOR AFFINE AND METRIC SPACES

BY JACK LEVINE

1. *Introduction.* Complete sets of identities have been obtained for the components of the affine and metric normal tensors and also for the components of the affine, metric, and projective curvature tensors. In addition to these identities, a complete set is known for the components of the first covariant derivative of the affine curvature tensor.*

In this paper a complete set of identities is obtained for the components of the second covariant derivative of the affine curvature tensor, and also a complete set for the first and second covariant derivatives of the metric curvature tensor.

2. *The First Covariant Derivative.* We shall now obtain a complete set of identities for the components† $B_{ijkl,m}$ of the first covariant derivative of the metric curvature tensor with components B_{ijkl}. In terms of the g_{ij} and their derivatives, we have

$$(1)\qquad B_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) + g_{ab}(\Gamma^a_{ik}\Gamma^b_{jl} - \Gamma^a_{il}\Gamma^b_{jk}).$$

By expressing (1) in metric normal coordinates, differentiating and evaluating at the origin, we obtain

$$(2)\qquad B_{ijkl,m} = (g_{ik,jlm} + g_{jl,ikm} - g_{il,jkm} - g_{jk,ilm})/2.$$

The quantities $g_{ij,klm}$ are the components of the third extension of the g_{ij}. The following identities form a complete set for these components:

$$(3)\qquad g_{ij,klm} = g_{ji,klm} = g_{ij,pqr},$$
$$g_{ij,klm} + g_{ik,jlm} + g_{il,jkm} + g_{im,jkl} = 0,$$

---

* T. Y. Thomas, *Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934, pp. 81, 114, 132, 138.

† In this section and the next small Latin indices have the range $1, 2, \cdots, n$.

where pqr is any permutation of klm. From (2) and (3), it is not difficult to show that

$$
\begin{aligned}
&B_{ijkl,m} = -B_{jikl,m} = -B_{ijlk,m},\\
&B_{ijkl,m} + B_{iljm,k} + B_{ijmk,l} = 0,\\
&B_{ijkl,m} + B_{iklj,m} + B_{iljk,m} = 0.
\end{aligned} \tag{4}
$$

The equation (2) can be solved for the $g_{ij,klm}$, the solutions being

$$
\begin{aligned}
6g_{ij,klm} = {} & B_{ikjl,m} + B_{ikjm,l} + B_{iljm,k} + B_{jkil,m}\\
& + B_{jkim,l} + B_{jlim,k}.
\end{aligned} \tag{5}
$$

In obtaining (5) we use the identities (3) and also

$$g_{kl,ijm} = g_{ij,klm} + g_{jm,kli} + g_{mi,klj}, \tag{6}$$

this latter identity being a consequence of (3). Starting with (5) and (4), it is possible to obtain (2) and (3), and hence we have the following theorem.

THEOREM 1. *The identities* (4) *constitute a complete set of identities for the components* $B_{ijkl,m}$ *of the covariant form of the first covariant derivative of the curvature tensor of a metric space.*

3. *The Second Covariant Derivative.* We now consider the complete sets of identities for the components of the second covariant derivatives of the affine and metric curvature tensors. The affine case is first discussed. In terms of the components of the affine normal tensors, we have

$$B^{i}_{jkl,m,p} = A^{i}_{jklmp} - A^{i}_{jlkmp} + \psi^{i}_{jklmp}, \tag{7}$$

where

$$
\begin{aligned}
\psi^{i}_{jklmp} = {} & A^{i}_{alm}A^{a}_{jkp} + A^{i}_{alp}A^{a}_{jkm} + A^{i}_{amp}A^{a}_{jkl} + A^{i}_{alk}A^{a}_{jmp}\\
& + A^{i}_{jla}A^{a}_{kmp} + A^{i}_{ajk}A^{a}_{lmp} - (A^{i}_{akp}A^{a}_{jlm} + A^{i}_{akm}A^{a}_{jlp}\\
& + A^{i}_{amp}A^{a}_{jlk} + A^{i}_{akl}A^{a}_{jmp} + A^{i}_{jal}A^{a}_{kmp} + A^{i}_{jka}A^{a}_{lmp}).
\end{aligned} \tag{8}
$$

A complete set of identities for the components A^{i}_{jklmp} is given by the formulas

$$A^{i}_{jklmp} = A^{i}_{kjlmp} = A^{i}_{jkrst}, \quad (rst \text{ is any permutation of } lmp). \tag{9}$$

$$
\begin{aligned}
&A^{i}_{jklmp} + A^{i}_{jlkmp} + A^{i}_{jmlkp} + A^{i}_{jplmk} + A^{i}_{klmpj} + A^{i}_{kmlpj}\\
&\quad + A^{i}_{kpmlj} + A^{i}_{lmpjk} + A^{i}_{lpmjk} + A^{i}_{mpjkl} = 0.
\end{aligned} \tag{10}
$$

By differentiating covariantly the components $B^i_{jkl,m}$ in a complete set of identities which these components satisfy, we obtain

$$(11)\qquad \begin{gathered} B^i_{jkl,m,p} = -B^i_{jlk,m,p}, \quad B^i_{jkl,m,p} + B^i_{klj,m,p} + B^i_{ljk,m,p} = 0, \\ B^i_{jkl,m,p} + B^i_{jlm,k,p} + B^i_{jmk,l,p} = 0. \end{gathered}$$

If in (7) the indices m and p be interchanged and the resulting equation be subtracted from (7), there results

$$(12)\qquad \begin{aligned} B^i_{jkl,m,p} - B^i_{jkl,p,m} = {} & B^a_{jkl}B^i_{amp} - B^i_{akl}B^a_{jmp} \\ & - B^i_{jal}B^a_{kmp} - B^i_{jka}B^a_{lmp}. \end{aligned}$$

This latter identity, (12), is one of Ricci's identities. Use is made of the relations

$$(13)\qquad B^i_{jkl} = A^i_{jkl} - A^i_{jlk}$$

in obtaining it. The A^i_{jklmp} can be obtained in terms of the B's by use of (7), (9), (10), the result being

$$(14)\qquad \begin{aligned} 10A^i_{jklmp} = {} & 2\lambda^i_{jklmp} + \lambda^i_{jkmlp} + \lambda^i_{jkplm} + \lambda^i_{kjlmp} + \lambda^i_{kjmlp} \\ & + \lambda^i_{kjpml} + \lambda^i_{ljmkp} + \lambda^i_{ljpmk} + \lambda^i_{jklpm} \\ & + \lambda^i_{mjpkl} + \lambda^i_{jkmpl}, \end{aligned}$$

where we have put

$$(15)\qquad \lambda^i_{jklmp} = B^i_{jkl,m,p} - \psi^i_{jklmp}.$$

In (15) the ψ are to be thought of as being expressed in terms of the B^a_{bcd}, this being done with the aid of the formula

$$(16)\qquad A^a_{bcd} = (B^a_{bcd} + B^a_{cbd})/3.$$

The components B^a_{bcd} satisfy the following identities which form a complete set:

$$(17)\qquad B^a_{bcd} = -B^a_{bdc}, \quad B^a_{bcd} + B^a_{cdb} + B^a_{dbc} = 0.$$

From (11), (12), (17), and (14) it can be shown after rather lengthy calculations that the complete set of identities (9), (10) is satisfied, and also the equations (7) can be obtained. Hence we have the following theorem.

THEOREM 2. *The identities* (11), (12), (17) *constitute a complete set of identities for the components* $B^i_{jkl,m,p}$ *and* B^i_{jkl}.

The identities for the components $B_{ijkl,m,p}$ of the second covariant derivative of the metric curvature tensor with components B_{ijkl} will now be considered. In terms of the components of the metric normal tensors and the g_{ij} and their extensions it is found that

$$B_{ijkl,m,p} = g_{ia}(A^{a}_{jklmp} - A^{a}_{jlkmp}) + h_{ijklmp}, \tag{18}$$

where

$$\begin{aligned} h_{ijklmp} = g_{ia}(&A^{b}_{jkp}A^{a}_{blm} + A^{a}_{blp}A^{b}_{jkm} - A^{b}_{jlp}A^{a}_{bkm} \\ &- A^{a}_{bkp}A^{b}_{jlm} + B^{b}_{jkl}A^{a}_{bmp}) \\ &- B_{ibkl}A^{b}_{jmp} - B_{ijbl}B^{b}_{kmp} - B_{ijkb}B^{b}_{lmp}. \end{aligned} \tag{19}$$

Also

$$B_{abcd} = g_{ac,bd} - g_{bc,ad}, \tag{20}$$

and A^{a}_{bcd} can be obtained in terms of the g's by means of (16) and (20). We find that

$$\begin{aligned} &g_{dk}(A^{k}_{ijlmp} - A^{k}_{iljmp}) \\ &= (1/2)(g_{dj,ilmp} + g_{il,djmp} - g_{dl,ijmp} - g_{ij,dlmp}) + a_{dijlmp}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} a_{dijlmp} = &A^{b}_{ilm}g_{bd,jp} - A^{b}_{ijm}g_{bd,lp} + A^{b}_{ilp}g_{bd,jm} \\ &- A^{b}_{ijp}g_{bd,lm} - R^{b}_{ijl}g_{bd,mp}, \end{aligned} \tag{22}$$

and $g_{ij,klmp}$ are the components of the fourth extension of the g_{ij}. If we place

$$C_{ijklmp} = a_{ijklmp} + h_{ijklmp}, \tag{23}$$

then (18) can be rewritten, by means of (21),

$$\begin{aligned} B_{ijkl,m,p} = &(1/2)(g_{ik,jlmp} + g_{jl,ikmp} - g_{il,jkmp} - g_{jk,ilmp}) \\ &+ C_{ijklmp}. \end{aligned} \tag{24}$$

It is evident that C_{ijklmp} can be expressed in terms of the g's. The $g_{ij,klmp}$ satisfy the complete set of identities

$$\begin{aligned} &g_{ij,klmp} = g_{ji,klmp} = g_{ij,qrst}, \ (qrst \text{ is any permutation of } klmp), \\ &g_{ij,klmp} + g_{ik,jlmp} + g_{il,jkmp} + g_{im,jklp} + g_{ip,jklm} = 0. \end{aligned} \tag{25}$$

From (4) and (12) we can obtain the identities

$$
\begin{aligned}
&B_{ijkl,m,p} = -B_{jikl,m,p} = -B_{ijlk,m,p},\\
&B_{ijkl,m,p} + B_{iklj,m,p} + B_{iljk,m,p} = 0,\\
&B_{ijkl,m,p} + B_{ijlm,k,p} + B_{ijmk,l,p} = 0,\\
&B_{ijkl,m,p} - B_{ijkl,p,m} = B_{iamp}B^{a}_{jkl} - B_{iakl}B^{a}_{jmp}\\
&\qquad\qquad - B_{ijal}B^{a}_{kmp} - B_{ijka}B^{a}_{lmp}.
\end{aligned} \tag{26}
$$

It is found that we can solve for the components $g_{ij,klmp}$ by the use of (24) and (25), obtaining

$$
\begin{aligned}
10g_{ij,klmp} = {} & E_{ikjlmp} + E_{jkilmp} + E_{ikjmlp} + E_{jkimlp}\\
& + E_{ikjpml} + E_{jkipml} + E_{iljmkp} + E_{jlimkp}\\
& + E_{iljpmk} + E_{jlipmk} + E_{imjpkl} + E_{jmipkl},
\end{aligned} \tag{27}
$$

and

$$
E_{ijklmp} = B_{ijkl,m,p} - C_{ijklmp}. \tag{28}
$$

The following identities are useful in obtaining (27)

$$
\begin{aligned}
g_{mp,ijkl} = {} & g_{ij,klmp} + g_{ik,jlmp} + g_{il,jkmp} + g_{jk,ilmp}\\
& + g_{jl,ikmp} + g_{kl,ijmp},
\end{aligned} \tag{29}
$$

these being derivable from (25). The quantities E are expressed in terms of the g_{ij}, B_{ijkl}, $B_{ijkl,m,p}$. Now we can obtain (25) and (24) by means of (27) and (26), and as the B_{ijkl} have the complete set of identities

$$
B_{ijkl} = -B_{jikl} = -B_{ijlk};\ B_{ijkl} + B_{iklj} + B_{iljk} = 0, \tag{30}
$$

we have the following theorem.

THEOREM 3. *The identities* (26), (30) *and* $g_{ij}=g_{ji}$ *constitute a complete set of identities for the components* g_{ij}, B_{ijkl}, *and* $B_{ijkl,m,p}$.

The components g_{ij} have to be included since they appear explicitly in the right member of (27).

PRINCETON UNIVERSITY

ON THE LIMIT OF A SEQUENCE OF POINT SETS

BY HSÜ PAO-LU

A variable point P_n is said to approach the point P as its limit if to an arbitrary positive ϵ there corresponds an m such that

$$\overline{P_nP} < \epsilon, \qquad (n > m).$$

In other words, P is to have the property that every neighborhood of it contains almost all* the points P_n.

In attempting to generalize this definition to a sequence of point sets $M_1, M_2, \cdots$, one is naturally led to begin with a definition of the neighborhood of a set and then write down (Definition A_0) the last sentence of the last paragraph, replacing the letter P by M.

DEFINITION. *By the ϵ-neighborhood of a set M is meant the set of all points which have a distance $<\epsilon$ from some point of M. We shall denote it by $(\epsilon)_M$.*

DEFINITION A_0. *A point set M is called a limit of the sequence of sets $M_1, M_2, \cdots$, if every neighborhood of it contains almost all the sets M_i as partial sets.*

But the above definition is far from being useful, because the limit would then not be unique. In the first place, if the set M is a limit in the sense of Definition A_0, and if M has a cluster point C, then the set $M-C$ has also the property of being a limit of the sequence. Secondly every set containing M as a partial set is a fortiori a limit.

The first difficulty is overcome by requiring M to be closed, and the second difficulty is met by adding still another condition (γ):

DEFINITION A. *A set M is said to be the limit of the sequence of sets $M_1, M_2, \cdots$, if it has the following properties:*

(α) *M is closed.*

---

* Thereby is meant that at most a finite number of the points P_i can lie outside the neighborhood.

(β) *For an arbitrary* $\epsilon>0$, $(\epsilon)_M \supset M_i$* *for almost all indices* i.
(γ) *For an arbitrary* $\epsilon>0$, $(\epsilon)_{M_i} \supset M$ *for almost all indices* i.

Observe that in the case of a sequence of points, (α) is fulfilled, (β) and (γ) become equivalent, and the definition reduces to the old one.

The following are immediate consequences of the definition:

(1) If a sequence of sets has a limit, the limit is unique.

(2) If a sequence $\mathfrak{S}$ of point sets has the limit M, every partial sequence of $\mathfrak{S}$ has the same limit M.

Further results hereby obtained consist of two fundamental criteria for the existence of a limit, when we restrict the sets of the sequence to lying in the same finite region of space. Given a sequence $\mathfrak{S}$ of sets $M_1, M_2, \cdots$, an L-point of $\mathfrak{S}$ shall be defined as a point which is the limit of a sequence of points $P_1, P_2, \cdots$, where each P_i belongs to the set M_i.

THEOREM A. *Let*

$$\mathfrak{S}: \qquad M_1, M_2, \cdots$$

be a sequence of point sets such that all the M_i*'s lie in the same finite region of space. Then* $\mathfrak{S}$ *has a limit when and only when, whatever partial sequence* $\mathfrak{S}_1$ *be selected from* $\mathfrak{S}$, *the set of* L*-points of* $\mathfrak{S}_1$ *coincides with the set of* L*-points of* $\mathfrak{S}$. *The limit of* M_i *is then the set of* L*-points of* $\mathfrak{S}$.

THEOREM B. *A necessary and sufficient condition for the sequence of sets* M_i, *lying in the same finite region of space, to have a limit is that, to an arbitrary positive* ϵ, *there corresponds an* M_m *such that*

(β') $(\epsilon)_{M_m} \supset M_i$ *for almost all indices* i,
(γ') $(\epsilon)_{M_i} \supset M_m$ *for almost all indices* i.

EXAMPLE 1. If each M_i is closed, M_1 is bounded, and $M_i \supset M_{i+1}$ for all i's, then a limit M exists and is equal to the set of points common to all the M_i's.

EXAMPLE 2. If $M_{i+1} \supset M_i$ and all the M_i's lie in the same finite region of space, then a limit M exists and is equal to the closed cover† of the set of points which belong to one of the M_i's.

* Read: "the ϵ-neighborhood of M contains M_i as a partial set."

† The closed cover of a set is the sum of the set and its first derived set.

In the case where each M_i is a point, the meaning of Theorem A is obvious, while Theorem B leads directly to the fundamental criterion for the variable point P_n to approach a limit, namely, to an arbitrary $\epsilon>0$ there corresponds an m such that $\overline{P_nP_{n'}}<\epsilon$, provided that $n, n'<m$.

But what do these theorems tell us when the set M_i corresponds to a point function?

To be exact, consider a sequence of functions $f_1(P), f_2(P), \cdots$, defined in the same bounded set N of $(n-1)$-dimensional space, and converging toward a limiting function $f(P)$, in each point P of N. Moreover, let the functions $f_i(P)$, $f(P)$ be bounded; that is, $|f_i(P)|<G$, $|f(P)|<G$, where G is the same number for all the functions. To each f_i corresponds then a bounded set M_i, formed of the points $(x_1, x_2, \cdots, x_{n-1}, x_n)$, where $P:(x_1, x_2, \cdots, x_{n-1})$ is a point of N and $x_n=f_i(P)$. Furthermore, all the sets M_i lie in the same finite region of space. Let M be the set corresponding to f as M_i to f_i. The following result is immediate. *If f_i is uniformly convergent, M_i has the closed cover of M as its limit.* But the converse is not true. For example, let N be the interval (0, 1), and

$$f_i = \begin{cases} \epsilon_i & \text{when } 0 \leqq x \leqq \eta_i, \\ 1-\epsilon_i & \text{when } \eta_i < x \leqq 1, \end{cases}$$

where $\epsilon_i>0$, $\eta_i>0$, $\epsilon_i>\epsilon_{i+1}$, $\eta_i<\eta_{i+1}$, $\lim_{n=\infty}\epsilon_n=0$, $\lim_{n=\infty}\eta_n=1/2$. Here f_i converges non-uniformly, while M_i has the closed cover of M as its limit.

Nevertheless it is true that *if N is closed and f is continuous, then f_i converges uniformly when M_i approaches M as its limit.* Thus under appropriate restrictions Theorem B is equivalent to the condition for uniform convergence, namely, f_i converges uniformly when and only when, to an arbitrary $\epsilon>0$, there corresponds an m, independent of P, such that

$$|f_n(P)-f_{n'}(P)|<\epsilon, \qquad (m<n, n').$$

Under the same restrictions Theorem A may be translated as follows. *The function f_i is uniformly convergent when and only when, for every sequence of points $P_1, P_2, \cdots$ of N with the limit P, the sequence of numbers $f_1(P_1), f_2(P_2), \cdots$ has the limit $f(P)$.*

NATIONAL UNIVERSITY OF PEKING, PEIPING, CHINA

THE NUMBER SYSTEM AFTER DEDEKIND

BY W. F. OSGOOD

1. *Introduction.* The rational numbers (positive, negative, and zero) having been defined and the four species for them established, the creation of the irrational numbers is possible in a variety of ways, notably by *regular sequences* and by Dedekind's *cut.*

The first method is, perhaps, easiest for the beginner. The fundamental principle of limits once being accepted in the form of the *definition* of a regular sequence, the detailed developments proceed smoothly, and the system of real numbers is evolved satisfactorily.*

Complete as this method is, there are mathematicians who feel that the definition of irrational numbers by means of the cut is essentially simpler, more elementary, for it creates these numbers before the concept of a limiting process is introduced and thus provides the elements for point sets before any infinite process has been defined.

Addition and subtraction are defined in the simplest manner possible on the basis of the cut. If $A=(a_1, a_2)$ and $B=(b_1, b_2)$ are any two numbers, then $C=A+B$ is the number

$$(c_1, c_2) = (a_1 + b_1, a_2 + b_2).$$

But multiplication cannot be defined in the same way. One would like to define $C=AB$ as the number $(c_1, c_2)=(a_1b_1, a_2b_2)$; but this is impossible since the fractions a_1b_1, a_2b_2 do not yield a cut. True, one can introduce the positive irrationals after the positive fractions, and before the negative numbers and zero, have been defined. The above definition of multiplication as the product of two cuts is now possible. The definition of negative numbers goes through exactly as before, and the system of real numbers is complete. We have lost, however, the earlier definition of a real number as a cut, and this is too high a price to pay for the simplified definition of multiplication.

* Pierpont has given an excellent exposition of this method in his *Functions of Real Variables,* vol. 1, pp. 31–61.

2. *The Way Out.* There is, however, a way out, which is provided by the conception of the limit set forth in Mr. Hsü's paper.* It has already been remarked that as soon as the system of real numbers has been created through the definition of the cut, and $<$, $=$, $>$ have been defined, the basis for a theory of point sets has been provided, even before addition, to say nothing of multiplication, has been defined. In particular, it is possible to define the *limit* of a point set.

By the *neighborhood* of a point ($=$*number*), $A=(a_1, a_2)$, shall be meant the numbers x such that

$$\gamma_1 < x < \gamma_2,$$

where γ_1 is a number $<A$ and γ_2 is a number $>A$.

By the ϵ-*neighborhood* of A shall be meant the neighborhood for which

$$\gamma_1=(a_1-\epsilon,\ a_2-\epsilon), \qquad \gamma_2=(a_1+\epsilon,\ a_2+\epsilon),$$

where ϵ is any positive fraction.

3. *Definition of Convergence.* Let $M_1, M_2, \cdots$ be a succession, or sequence, of point sets. Then this sequence is said to *converge* if to an arbitrary fraction $\epsilon>0$ there corresponds an integer m and a point ζ_m of the set M_m such that the ϵ-neighborhood of ζ_m contains all the sets M_n for which $m \leqq n$.

THEOREM 1. *Let $M_1, M_2, \cdots$ be a convergent sequence. Then there exists one and only one point*

$$u=(u_1,\ u_2)$$

such that an arbitrary ϵ-neighborhood of U contains each M_n, $m \leqq n$; and conversely.

4. *Definition of a Limit.* Let $M_1, M_2, \cdots$ be a convergent sequence. The number U related to it by Theorem 1 is defined as the *limit* of the sequence:

$$U = \lim_{n=\infty} M_n.$$

THEOREM 2. *Let $M_1, M_2, \cdots$ be a convergent sequence. Let N_n be composed of some or all the points of M_n. Then the sequence $N_1, N_2, \cdots$ is convergent, and its limit is equal to the limit of the sequence $M_1, M_2, \cdots$.*

* This Bulletin, vol. 41 (1935), p. 502.

5. *Definition of Multiplication.* Let $A=(a_1, a_2)$ and $B=(b_1, b_2)$ be any two numbers. Let

$$\epsilon_1 \geqq \epsilon_2 \geqq \cdots, \qquad \lim_{n=\infty} \epsilon_n = 0;$$

$$\eta_1 \geqq \eta_2 \geqq \cdots, \qquad \lim_{n=\infty} \eta_n = 0,$$

be two sequences of positive fractions. Let M_n be the point set composed of the fractions $a_p b_q$,

$$M_n: \qquad \{a_p b_q\},$$

where a_p, $(p=1, 2)$, is any fraction of the ϵ_n-neighborhood of A, and b_q, $(q=1, 2)$, is any fraction of the η_n-neighborhood of B. Then the sequence $M_1, M_2, \cdots$ converges, and its limit is defined as the *product* AB:

$$AB = \lim_{n=\infty} \{a_p b_q\}.$$

That multiplication obeys the commutative law, $AB=BA$, is at once obvious. The associative law is easily established by means of the sequence $M_1', M_2', \cdots$, where $C=(c_1, c_2)$ is any third number and M_n' is composed of the fractions $a_p b_q c_r$,

$$M_n': \qquad \{a_p b_q c_r\},$$

c_r, $(r=1, 2)$, being any fraction lying in the λ_n-neighborhood of C. Here,

$$\lambda_1 \geqq \lambda_2 \geqq \cdots, \qquad \lim_{n=\infty} \lambda_n = 0,$$

is a sequence of positive fractions. The sequence $M_1', M_2', \cdots$ converges; and now it is shown without difficulty that its limit has the value $(AB)C$. But the limit is obviously invariant of the order of the factors a_p, b_q, c_r. Hence it also has the value

$$(BC)A = A(BC).$$

Division is now shown to be possible and unique, except when the divisor is zero.

THE NATIONAL UNIVERSITY OF PEKING

FURTHER NON-INVOLUTORIAL CREMONA SPACE TRANSFORMATIONS CONTAINED IN A SPECIAL LINEAR COMPLEX*

BY AMOS BLACK

1. *Introduction.* In a series of papers by Snyder,† Carroll,‡,§ and the author, ‖,¶ involutorial transformations were defined by means of a correspondence between the surfaces of a pencil and the points of a rational curve. The purpose of this paper is to apply similar methods to certain non-involutorial transformations.

2. *Definition of the Transformation.* Given a line d and two pencils of surfaces $|F_n|$ and $|F_{n'}'|$ of orders n and n' which contain d as an $(n-1)$-fold and $(n'-1)$-fold line, respectively. Make the surfaces of each pencil projective with the points of d. A point P will determine a unique surface F_n passing through it, hence a unique point O on d and a unique surface $F_{n'}'$. The line PO cuts $F_{n'}'$ in one point P' (other than O) which is defined as the image of P.

Since P and P' lie on a line which intersects d, any plane through d is transformed into itself. We shall find the plane transformation in an arbitrary plane through d and then generate the space transformation by revolving the plane about d.

3. *The Plane Transformation.* The intersections of an arbi-

* Presented to the Society, December 27, 1933.

† Virgil Snyder, *On a series of involutorial cremona transformations of space defined by a pencil of ruled surfaces*, Transactions of this Society, vol. 35 (1933), pp. 341–347.

‡ Evelyn Carroll, *Systems of involutorial birational transformations contained multiply in special linear line complexes*, American Journal of Mathematics, vol. 54 (1932), pp. 707–717.

§ Evelyn Carroll-Rusk, *Cremona involutions defined by a pencil of cubic surfaces*, American Journal of Mathematics, vol. 56 (1934), pp. 96–108.

‖ Amos Black, *Types of involutorial space transformations associated with certain rational curves*, Transactions of this Society, vol. 34 (1932), pp. 795–810.

¶ Amos Black, *Types of involutorial space transformations associated with certain rational curves—composite basis curves*, this Bulletin, vol. 40 (1934), pp. 417–420.

trary plane through d with $|F_n| : d^{n-1}$ and $|F'_{n'}| : d^{n'-1}$ are two pencils of lines $|l| : V$ and $|l'| : V'$, respectively, where V and V' are the vertices of the pencils. The lines of each pencil are projective with the points of d. Thus a point P determines a line l through it, a point O on d, and a line l'. The point P', the image of P, is the intersection of PO and l'.

The directions through V are perspective with the points of d which are projective with the lines $|l'|$. The conic c_2' of intersection of these two projective pencils is the image of V.

Since there is a (1, 1) correspondence between the lines of $|l|$ and the points of d, there are two coincidences, hence two lines l_3, l_4 which pass through their respective associated points O_1, O_2. Let us call the associated lines of $|l'|$, l_1' and l_2', respectively. If we choose P as an arbitrary point on l_3, its image is P_1', the point of intersection of l_3 and l_1'. If we choose P at O_1 the line PO_1 is indeterminate. However, we may readily find the image of O_1 from the inverse transformation. If we choose P' as an arbitrary point on l_1', then $P'O_1$ intersects l_3 in O_1. Thus O_1 has for image the whole line l_1'. Hence $l_3 : VO_1P_1' \sim c_2' l_1' P_1'$. Similarly $l_4 : VO_2P_2' \sim c_2' l_2' P_2'$. Beginning with V' and $|l'|$, in a similar manner we find $l_3' : V'O_1'P_1 \sim c_2 l_1 P_1$ and $l_4' : V'O_2' P_2 \sim c_2 l_2 P_2$.

There are no other fundamental points or curves, hence the plane transformation is the well known $T_3 : 1^2 4^1$.

$$
\begin{aligned}
&T_3 : V^2 O_1 O_2 P_1 P_2, && T_3' : V'^2 O_1' O_2' P_1' P_2',\\
&J_6 = c_2 l_1 l_2 l_3 l_4, && J_6' = c_2' l_1' l_2' l_3' l_4',\\
&V \sim c_2' : VV'O_1'O_2'P_1'P_2', && V' \sim c_2 : VV'O_1O_2P_1P_2,\\
&O_1 \sim l_1' : V'P_1', && O_1' \sim l_1 : VP_1,\\
&O_2 \sim l_2' : V'P_2', && O_2' \sim l_2 : VP_2,\\
&P_1 \sim l_3' : V'O_1' P_1, && P_1' \sim l_3 : VO_1P_1',\\
&P_2 \sim l_4' : V'O_2' P_2, && P_2' \sim l_4 : VO_2P_2'.
\end{aligned}
$$

The invariant curve is the conic k_2, the intersection of the two projective pencils $|l|$, $|l'|$. Hence $k_2 : VV'P_1P_2P_1'P_2'$.

4. *The Space Transformation.* Since the space transformation is generated by the plane T_3, except for orders, we can immediately determine the surfaces of the space transformation. As

we revolve the plane of T_3 about d, the points O_1, O_2 generate d. Hence l_1', l_2' generate a surface L' which is the image of d. The points P_1, P_2 generate a curve δ, whose image X' is the locus of l_3', l_4'. The vertex V generates the residual basis curve γ of the pencil $|F_n|$, and the image Γ' of γ is the locus of c_2'. In a similar manner δ', L, X, Γ are obtained. The locus of k_2 is the invariant surface K.

Since the sections of all the surfaces of the space transformation with a plane through d are known, except for the multiplicity of d, whatever the values of n and n' may be, there is no advantage in having n and n' large. We shall set up the analytic work for $n=n'=2$.

5. *Equations of the Transformation.* Let the equations of d be $x_1=0, x_2=0$, and let the coordinates of a point on d be given by $O(0, 0, \lambda, 1)$. Also let

$$|H(x)| = H_1(x) - \lambda H_2(x) = 0, \tag{1}$$

$$|H'(x)| = H_1'(x) - \lambda H_2'(x) = 0, \tag{2}$$

where

$$H_1(x) = x_1u_1(x) + x_2u_2(x), \qquad H_2(x) = x_1v_1(x) + x_2v_2(x),$$

$$H_1'(x) = x_1u_1'(x) + x_2u_2'(x), \qquad H_2'(x) = x_1v_1'(x) + x_2v_2'(x),$$

$$u_i(x) = \sum a_{ij}x_j, \qquad v_i(x) = \sum b_{ij}x_j,$$

$$u_i'(x) = \sum a_{ij}'x_j, \qquad v_i'(x) = \sum b_{ij}'x_j,$$

for $i=1, 2$ and $j=1, 2, 3, 4$. Since the parameters of the points of d and the surfaces of (1) and (2) are identical, the forms are projective.

The surface of (1) passing through $P(x)$ gives $\lambda=H_1/H_2$, where $H_1=H_1(x)$ and $H_2=H_2(x)$, and the associated point on d is $O(z)=(0, 0, H_1, H_2)$, and the associated surface of (2) is

$$H_2H_1'(x) - H_1H_2'(x) = 0. \tag{3}$$

The coordinates of any point on the line PO are given by

$$x_i' = \rho x_i + z_i, \qquad (i = 1, 2, 3, 4). \tag{4}$$

Substituting from (4) in (3) gives $\rho=L_5/K_4$, where

$$
\begin{aligned}
L_5 &= H_2^2(a'_{14}x_1 + a'_{24}x_2) + H_1H_2(a'_{13}x_1 + a'_{23}x_2 \\
&\quad - b'_{14}x_1 - b'_{24}x_2) - H_1^2(b'_{13}x_1 + b'_{23}x_2), \\
K_4 &= H_2H'_1 - H_1H'_2 .
\end{aligned} \tag{5}
$$

The equations of the transformation are

$$
T_6: \quad x'_i = x_iL_5 - K_4z_i, \qquad (i = 1, 2, 3, 4). \tag{6}
$$

It is evident from the form of the equations of T_6 that $K_4 = 0$ is the surface of invariant points, and that $L_5 = 0$ is the image of d.

Any arbitrary plane π through d is the tangent plane of two surfaces of $|H'(x)|$ at points on d. The two points of contact are the points O'_1, O'_2 of the plane T_3 in π. The intersections of π with the two surfaces of $|H'(x)|$ associated with O'_1, O'_2 are the lines l'_3, l'_4 and the intersections with the associated surfaces of $|H(x)|$ are the lines l_1, l_2. Since L_5 is the locus of l_1, l_2, then the elimination of λ between the equations of the tangent plane of a surface $H'(x)$ at its associated point O and the associated surface $H(x)$ gives L_5.

The tangent plane of $H'(x)$ at its associated point O is

$$
\begin{aligned}
&[a'_{13}\lambda + a'_{14} - \lambda(b'_{13}\lambda + b'_{14})]x_1 \\
&\quad + [a'_{23}\lambda + a'_{24} - \lambda(b'_{23}\lambda + b'_{24})]x_2 = 0.
\end{aligned} \tag{7}
$$

The elimination of λ between (7) and (1) gives L_5. Similarly, since the locus of l'_3, l'_4 is X'_5, then the elimination of λ between (7) and (2) gives X'_5, or

$$
\begin{aligned}
X'_5 &= H_2'^2(a'_{14}x_1 + a'_{24}x_2) + H'_1H'_2(a'_{13}x_1 + a'_{23}x_2 \\
&\quad - b'_{14}x_1 - b'_{24}x_2) - H_1'^2(b'_{13}x_1 + b'_{23}x_2).
\end{aligned} \tag{8}
$$

Since any quadric of $|H'(x)|$ is transformed into the associated quadric of $|H(x)|$, the equation of the image of γ'_3 may be found by transforming $H'_1(x)$. Hence $H'_1(x) \sim H_1L_5\Gamma_5$, where

$$
\begin{aligned}
\Gamma_5 &= [H'_2(a'_{14}x_1 + a'_{24}x_2) - H'_1(b'_{14}x_1 + b'_{24}x_2)]H_2 \\
&\quad + [H'_2(a'_{13}x_1 + a'_{23}x_2) - H'_1(b'_{13}x_1 + b'_{23}x_2)]H_1.
\end{aligned} \tag{9}
$$

In a similar manner we may write the equations of

$$
T'_6: \quad x_i = x'_iL'_5 + K_4z'_i, \qquad (i = 1, 2, 3, 4), \tag{10}
$$

where $(z'_i) = (0, 0, H'_1, H'_2)$ and L'_5, X_5, Γ'_5 are found from L_5, X'_5, Γ_5 by interchanging primes and unprimes.

The multiplicates of γ_3, γ_3', δ_8, δ_8' on the various surfaces are known from T_3, and the multiplicity of d can easily be verified from equations (5) to (10). Of the three tangent planes of any homoloid at any point on d, one tangent plane is the tangent plane of the associated quadric at the point. Thus the homoloidal surfaces have simple contact along d. Collecting these results, we have

$$\begin{aligned}
\text{a plane} &\overset{T}{\sim} S_6 : d^{3+1t}\gamma_3^2\delta_8, & \text{a plane} &\overset{T'}{\sim} S_6' : d^{3+1t}\gamma_3'^2\delta_8', \\
d &\sim L_5 : d^3\gamma_3^2\delta_8, & d &\sim L_5' : d^3\gamma_3'^2\delta_8', \\
\gamma_3' &\sim \Gamma_5 : d^3\gamma_3\gamma_3'\delta_8, & \gamma_3 &\sim \Gamma_5' : d^3\gamma_3\gamma_3'\delta_8', \\
\delta_8' &\sim X_5 : d^3\gamma_3^2\delta_8', & \delta_8 &\sim X_5' : d^3\gamma_3'^2\delta_8, \\
& K_4 : d^2\gamma_3\gamma_3'\delta_8\delta_8'.
\end{aligned}$$

6. *Special Cases.* A pencil of quadrics whose base is a line and a twisted cubic contains two cones. In general, the vertex of the cone is not at its associated point in our projectivity. Whenever the vertex and the associated point coincide we have a special case.

CASE 1. Suppose $b_{13}' = b_{23}' = 0$. Then $H_2'(x)$ is a cone with vertex $O_1'(0, 0, 1, 0)$, and from (2), we see that the vertex and associated point coincide. In the pencil $|l'|$ of the plane T_3, one line is a generator of the cone, hence passes through its associated point O_1'. The image of O_1' is the line l_1 of H_2 lying in this plane. The image of the generator is P_1, the point of intersection of the generator and l_1. But as the plane is rotated about d, the point O_1' remains fixed, l_1 generates H_2, and P_1 generates a twisted cubic δ_3. Thus, in space, $O_1' \sim H_2$, and $H_2' \sim \delta_3$. However, the other incidence point O_2' varies and generates d, and P_2 varies and generates a rational curve δ_5. It is easily seen from (5) that if $b_{13}' = b_{23}' = 0$, then $L_5 = H_2 L_3$, and from (8) that $X_5' = H_2' X_3'$. Then, in this special sense, in T_6' an isolated fundamental point O_1' with image H_2 is added, and in T_6 the fundamental curve δ_8 is composite with $\delta_3 \sim H_2'$ and $\delta_5 \sim X_3'$.

CASE 2. If a cone of $|H'(x)|$ has its vertex at its associated point and also a cone of $|H(x)|$ has its vertex at its associated point, but the vertices are distinct, then we have the specialization of Case 1 in both transformations. That is, both T_6 and T_6' have an isolated fundamental point on d, and both δ_8 and δ_8' are composite.

CASE 3. Let $|H(x)|$ and $|H'(x)|$ both have a cone with vertex at its associated point, and let the vertices coincide. Let the associated point be $O(0, 0, 1, 0)$ and let the cones be H_2 and H_2'. Then $b_{13}=b_{23}=b_{13}'=b_{23}'=0$. In the plane T_3 this means that O_1, O_1', P_1, P_1' all coincide with O and that $l_1=l_3$, $l_1'=l_3'$. In space $L_5=H_2L_3$, $X_5=H_2X_3$, $L_5'=H_2'L_3'$, $X_5'=H_2'X_3'$.

Since O_2, O_2', P_2, P_2', V, V' are variable, then as the plane of T_3 is revolved about d, these points generate d, d', δ_5, δ_5', γ_3, γ_3', respectively, and their respective images are L_3', L_3, X_3', X_3, Γ_5', Γ_5. The point O is fixed and $O \overset{T}{\sim} H_2'$ and $O \overset{T'}{\sim} H_2$. The two cones H_2 and H_2' intersect in d and in three other lines l_i, $(i=1, 2, 3)$. Each of these lines is parasitic; hence $\delta_3=\delta_3'=3l$.

CASE 4. Let $|H'(x)|$ have two cones with their vertices at their associated points. Let the two cones be H_2' with vertex O_1' $(0, 0, 1, 0)$ and H_1' with vertex O_2' $(0, 0, 0, 1)$. Then we have $b_{13}'=b'_3=a_{14}'=a_{24}'=0$. We find $L_5=H_1H_2\pi$, $X_5'=H_1'H_2'\pi$, where $\pi=a_{13}'x_1+a_{23}'x_2-b_{14}'x_1-b_{24}'x_2$. Then the equations of T_6 may be written in the form

$$x_i' = x_iH_1H_2\pi - K_4z_i, \qquad (i = 1, 2, 3, 4). \tag{11}$$

As we would expect from Case 1, $O_1' \sim H_2$, $O_2' \sim H_1$, $H_2' \sim \delta_3$, $H_1' \sim \bar{\delta}_3$.

As we rotate the plane T_3 about d the points O_1', O_2' remain fixed, so apparently no other points of d are fundamental, and hence d has no image; yet from the form of equations (11) the plane π is the image of d. The plane π is the tangent plane of every surface of $|H'(x)|$ at its associated point. Hence in the plane T_3 in plane π, every line of $|l'|$ passes through its associated point. Therefore, in plane π, $d \overset{T'}{\sim} \pi$ and $\pi \overset{T}{\sim} \delta_2$, where δ_2 is the conic of intersection of the two projective pencils in π.

Then in T_6' we introduce two isolated fundamental points, O_1' with image H_2 and O_2' with image H_1, and $d \sim \pi$; while in T_6 the curve δ_8 becomes composite, consisting of δ_3, $\bar{\delta}_3$, δ_2 with images H_2', H_1', π, respectively.

We may have combinations of Cases 1, 3, 4 just as Case 2 was a combination of a Case 1 with a Case 1.

7. *Special Cases due to Composite Basis Curves.* The only way in which γ_3 and γ_3' can be composite is that the composite curve consist of a line, or lines, meeting d and a residual curve.

CASE 5. Let $\gamma_3=\gamma_1\gamma_2$, where $\gamma_1: x_1=0$, $x_2=0$ and $[\gamma_1, d]=(0, 0, 0, 1)$. Then $a_{22}=b_{22}=a_{24}=b_{24}=0$, and $\Gamma_5'=y_1\Gamma_4'$, $X_5=y_1X_4$. No other surfaces of either T_6 or T_6' are composite.

Let us consider the plane T_3 in the plane $y_1=0$. The lines γ_1, d form the complete intersection of any H of $|H(x)|$ with the plane. Call the associated point of the composite H, of which y_1 is a component, $\bar{O}$. Fix a point P on γ_1. The line $P\bar{O}$ meets $l_{\bar{O}}'$ in a point P', the image of P. As $\bar{O}$ traces d, P' generates a conic $c_2: PV'\bar{O}_1'\bar{O}_2'$. As P traces γ_1, c_2 generates the plane y_1. Hence $\gamma_1\sim$ the plane. Consider an arbitrary point Q of the plane. The H determined by Q is composite and $\bar{O}$ is the associated point. The line $Q\bar{O}$ meets $l_{\bar{O}}'$ in Q', the image of Q. Then the whole line $Q\bar{O}$ transforms into Q', and the whole plane transforms into $l_{\bar{O}}'$. Thus the curve δ_8' is composite and consists of $\delta_1'=l_{\bar{O}}'$ and a δ_7'.

If γ_3 consists of a line γ_1 meeting d and a conic γ_2, then necessarily δ_8' consists of a δ_7', and a line δ_1' in the plane of γ_1, d; and the plane is a principal surface for both T_6 and T_6'.

CASE 6. If γ_3 consists of two lines each meeting d, and a third line skew to d, then the two planes determined by the two lines and d are principal for both T_6 and T_6', and δ_8' consists of a δ_6' and two lines, one in each of the principal planes.

CASE 7. If $\gamma_3=\gamma_1\gamma_2$ and $\gamma_3'=\gamma_1'\gamma_2'$, but γ_1 and γ_1' lie in different planes, then each plane is principal in both T_6 and T_6', and $\delta_8'=\delta_7'\delta_1'$ and $\delta_8=\delta_7\delta_1$.

CASE 8. If $\gamma_3=\gamma_1\gamma_2$, $\gamma_3'=\gamma_1'\gamma_2'$ and γ_1, γ_1' lie in the same plane but meet d in distinct points, then the result is the same as in Case 7 except that $\gamma_1=\delta_1$ and $\gamma_1'=\delta_1'$.

The restriction that γ_1, γ_1' meet d in distinct points is not necessary. If they meet d in the same point, or even coincide, the associated points $\bar{O}$, $\bar{O}'$ of the composite quadrics are, in general, distinct, hence the results are the same.

There may be special cases due to combinations of cases of §6 with cases of §7, but nothing new is added.

The line d may be replaced by a conic r_2 or a twisted cubic r_3 with the restriction that transformations exist for $n=n'=2$ only. However, the transformations are of the same form as the above, and no new results are obtained.

CORNELL UNIVERSITY

SPACE INVOLUTORIAL TRANSFORMATIONS OF THE GEISER AND BERTINI TYPES*

BY L. A. DYE

1. *Introduction.* One form of generalization of a plane involution is a space involutorial transformation in which each plane of a pencil is invariant and in each such plane there is a plane involution of the same type. Particular examples of this for Geiser and Bertini involutions have been given by Carroll,† Snyder and Lehr,‡ and Sharpe and Dye.§ I shall discuss a more general form for space involutorial transformations arising from these plane involutions by means of a mapping on a cubic surface.‖ The Bertini transformation obtained has the signature

$$I_{120n+51}: l^{120n+34+6t} + (O, \bar{O})^{120n+40} + C^{6}_{12n+6};$$

the Geiser transformation has the signature

$$I_{24n+19}: l^{24n+11+3t} + O^{24n+14} + C^{3}_{12n+6}.$$

2. *The Geiser Transformation.* In an involutorial space transformation of the Geiser type, let $x_4 = \lambda x_3$ be the equation of the invariant pencil of planes, and let I_G be the Geiser involution in the plane $x_4 = \lambda x_3$. Choose one of the fundamental points of I_G as $O \equiv (1, \lambda, 0, 0)$ on the line $l \equiv x_3 = x_4 = 0$, and map the I_G on a cubic surface F_3 by means of the bilinear $T_{3,3}$ defined by the matrices

$$(1) \qquad \left\| (a_{i1}y_i) \ (a_{i2}y_i) \ (a_{i3}y_i) \ (a_{i4}y_i) \right\|,$$

$$(2) \qquad \left\| (a_{1i}x_i) \ (a_{2i}x_i) \ (a_{3i}x_i) \ (a_{4i}x_i) \right\|,$$

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* Presented to the Society, December 27, 1934.

† E. T. Carroll, American Journal of Mathematics, vol. 54 (1931), pp. 707–717, and vol. 56 (1934), pp. 96–108.

‡ V. Snyder and M. Lehr, American Journal of Mathematics, vol. 53 (1931), pp. 186–195.

§ F. R. Sharpe and L. A. Dye, Transactions of this Society, vol. 36 (1934), pp. 292–305.

‖ For a discussion of the mapping of a Geiser or a Bertini plane involution on a cubic surface, see H. F. Baker, *Principles of Geometry*, vol. 6, pp. 122–130.

where here and in the rest of the paper only the first row of determinants and matrices is written; the second and third rows are obtained by replacing a_{ij} by b_{ij} and c_{ij}, respectively. To the plane $x_4=\lambda x_3$ corresponds the surface

$$F_3=\left|(a_{i1}y_i)\ (a_{i2}y_i)\ (a_{i3}'y_i)\right|=0, \tag{3}$$

where $a_{i3}'=a_{i3}+\lambda a_{i4}$. The point O goes into the point $P\equiv(D_i)$, where the D_i are the third-order determinants of the matrix $\|a_{11}'\ a_{21}'\ a_{31}'\ a_{41}'\|$, and where $a_{i1}'=a_{i1}+\lambda a_{i2}$. The image of the line l is the cubic curve $C_3\equiv\|(a_{i1}y_i)\ (a_{i2}y_i)\|=0$. The pairs of corresponding points of the Geiser involution J_G on the surface F_3 are determined by the pairs of intersections with F_3 of lines through P.

In the J_G on F_3 the image of P is the cubic curve cut from F_3 by the tangent plane p at P. The equation of p is

$$\left|(a_{i1}'y_i)\ (a_{i2}D_i)\ (a_{i3}'D_i)\right|=0. \tag{4}$$

The quadric cone H through C_3 with vertex at P has the equation

$$\left|(a_{i1}'y_i)\ (a_{i2}y_i)\ (a_{i2}D_i)\right|=0, \tag{5}$$

and meets F_3 in a residual cubic which is the image of C_3. The invariant curve of the J_G is cut from F_3 by the polar quadric of P with respect to F_3. This quadric K has for its equation

$$\left|(a_{i1}'y_i)\ (a_{i2}D_i)\ (a_{i3}'y_i)\right|+\left|(a_{i1}'y_i)\ (a_{i2}y_i)\ (a_{i3}'D_i)\right|=0. \tag{6}$$

If a_{ij}, b_{ij}, and c_{ij} are polynomials of order n in λ, and if λ is replaced by x_4/x_3, then the surfaces $y_i=0$ in the (x) space are of order $3n+3$. These surfaces have l as a $3n$-fold line and contain a curve C_{12n+6} of order $12n+6$, of genus $24n+3$, which meets l in $12n$ points. Since any plane through l is invariant under the Geiser space transformation, there is a pencil of homaloidal surfaces consisting of the pencil of planes through l and the images of l and O. The equations of these image surfaces are obtained by replacing y_i in the equations of H and p by the third-order determinants of the matrix (2). Since H, p, and K are of orders $12n+3$, $12n+7$, and $12n+4$ in λ, respectively, they correspond to surfaces of orders $12n+8$, $12n+10$, and $12n+10$.

The table of characteristics of the space involutorial transformation I_{24n+19} can now be written:

$$
\begin{aligned}
O &\sim F_{12n+10}: l^{12n+7+2t} + O^{12n+9} + C_{12n+6},\\
l &\sim F_{12n+8}: l^{12n+3+t} + O^{12n+4} + C^{2}_{12n+6},\\
C_{12n+6} &\sim F_{60n+44}: l^{60n+26+6t} + O^{60n+32} + C^{7}_{12n+6},\\
S_1 &\sim S_{24n+19}: l^{24n+11+3t} + O^{24n+14} + C^{3}_{12n+6},\\
K_{12n+10} &: l^{12n+4+2t} + O^{12n+6} + C^{2}_{12n+6},
\end{aligned}
$$

where the coefficient of t indicates the number of fixed tangent planes at a point of l. The image of the C_{12n+6} is obtained by applying the transformation to an S_{24n+19}.

The parasitic lines of the transformation consist of the trisecants of C_{12n+6} which meet l, and the bisecants of C_{12n+6} which pass through O. The number of trisecants of a C_m which meet a line l having i intersections with C_m is*

$$(m-2)[h - m(m-1)/6] - i(h - m + 2) + i(i-1)(i-2)/6,$$

where h is the number of apparent double points of C_m. The number of trisecants of C_{12n+6} which meet l is $24n+8$. In order to obtain the number of bisecants through O, it is necessary to set up a correspondence. Given a point μ on the line l, there are $h' = h - 12n(12n-1)/2 = 36n+7$ bisecants through it which determine h' planes λ. Given a plane λ through l there are six intersections of C_{12n+6} with it not on l and hence fifteen bisecants which determine fifteen points μ. In the (λ, μ) correspondence there are $h' + 15 = 36n + 22$ coincidences. There are then $36n+22$ bisecants of C_{12n+6} which meet O, and the total number of parasitic lines is $60n+30$.

Let ζ be the number of parasitic conics of the transformation, and let η be the number of parasitic cubics. The complete intersection of two surfaces of the web of S_{24n+19} is made up of

$$
\begin{aligned}
(24n+19)^2 = {} & 24n + 19 + (24n+11)^2 + 9 + 9(12n+6)\\
& + 60n + 30 + 8\zeta + 27\eta
\end{aligned}
$$

* L. A. Dye, this Bulletin, vol. 41 (1935), pp. 109–110.

curves, and the complete intersection of an S_{24n+19} and the K_{12n+10} is made up of

$$(24n+19)(12n+10) = 12n+10+(12n+4)(24n+11) + 6 + 6(12n+6) + 60n + 30 + 4\zeta + 9\eta$$

curves. The solution of these equations is $\zeta = 24n+16$, $\eta = 0$; therefore there are $24n+16$ conics and $60n+30$ lines of the second species in the I_{24n+19}.

3. *The Bertini Transformation.* The methods of the last section are now used to study an involutorial space transformation of the Bertini type. Let the invariant pencil of planes have the line $l \equiv x_3 = x_4 = 0$ as axis. In a plane $x_4 = \lambda x_3$ let $O \equiv (1, \mu, 0, 0)$ and $\overline{O} \equiv (1, -\mu, 0, 0)$, $(\mu^2 = \lambda)$, be two fundamental points of the Bertini involution I_B in the plane. The I_B is mapped on the cubic surface (3) by means of the $T_{3\text{-}3}$ defined by the matrices (1) and (2). The images of O and $\overline{O}$ are $P \equiv (D_i)$ and $\overline{P} \equiv (\overline{D}_i)$; D_i and $\overline{D}_i$ are the third-order determinants of the matrices $\|a'_{11}\ a'_{21}\ a'_{31}\ a'_{41}\|$, $\|\bar{a}'_{11}\ \bar{a}'_{21}\ \bar{a}'_{31}\ \bar{a}'_{41}\|$, where $a'_{i1} = a_{i1} + \mu a_{i2}$ and $\bar{a}_{i1} = a_{i1} - \mu a_{i2}$. The image of l is the C_3 given by the matrix equation $\|(a_{i1}y_i)\ (a_{i2}y_i)\| = 0$.

The tangent planes to F_3 at P, $\overline{P}$ are p, $\bar{p}$ and they have as equations

$$p \equiv |\ (a'_{i1}\ y_i)(\bar{a}'_{i1}\ D_i)\ (a'_{i3}D_i)\ | = 0,$$

$$\bar{p} \equiv |\ (a'_{i1}\ \overline{D}_i)(\bar{a}'_{i1}\ y_i)(a'_{i3}\ \overline{D}_i)\ | = 0.$$

We now define two numbers d, $\bar{d}$ as follows:

$$d \equiv p(\overline{D}_i) \equiv |\ (a'_{i1}\ \overline{D}_i)\ (\bar{a}'_{i1}D_i)\ (a'_{i3}D_i)\ |,$$

$$\bar{d} \equiv \bar{p}(D_i) \equiv |\ (a'_{i1}\ \overline{D}_i)\ (\bar{a}'_{i1}D_i)\ (a'_{i3}\overline{D}_i)\ |.$$

The residual intersection R of the line $P\overline{P}$ with F_3 has as coordinates $(\bar{d}D_i - d\overline{D}_i)$. The equation of the tangent plane to F_3 at R is

$$r \equiv d^2\bar{p} + \bar{d}^2 p - d\bar{d}\{|\ (a'_{i1}\ y_i)\ (\bar{a}'_{i1}D_i)\ (a'_{i3}\overline{D}_i)\ | + |\ (a'_{i1}\ \overline{D}_i)\ (\bar{a}'_{i1}\ y_i)\ (a'_{i3}D_i)\ | + |(a'_{i1}\ \overline{D}_i)\ (a'_{i1}D_i)(a'_{i3}\ y_i)\ |\} = 0.$$

The pairs of corresponding points in the Bertini involution J_B on F_3 are cut out by the conics tangent to F_3 at P and $\overline{P}$.

The image of P in J_B is the sextic cut from F_3 by the quadric having contact with F_3 at $\overline{P}$ and contact of the second order at P. Its equation is

$$H \equiv \bar{d}^2 d\big[\,|\,(a'_{i1}y_i)(\bar{a}'_{i1}D_i)(a'_{i3}y_i)\,| + |\,(a'_{i1}y_i)(\bar{a}'_{i1}y_i)(a'_{i3}D_i)\,|\,\big] + pr = 0;$$

similarly the quadric which determines the image of $\overline{P}$ has the equation

$$\overline{H} \equiv d^2\bar{d}\big[\,|\,(a'_{i1}\bar{D}_i)\ (\bar{a}'_{i1}y_i)\ (a'_{i3}y_i)\,| + |\,(a_{i1}y_i)\ (\bar{a}'_{i1}y_i)\ (a'_{i3}\bar{D}_i)\,|\,\big] + \bar{p}r = 0.$$

The cubic curve C_3 corresponds to a cubic cut from F_3 by the quadric through C_3 and touching F_3 at P and $\overline{P}$. This quadric has the equation

$$L \equiv |\,(a'_{i1}y_i)\ (a'_{i1}y_i)\ A\,| = 0,$$

where A, B, C are the second-order determinants of a two column matrix whose columns are made up of the second-order determinants of the matrix $\|(\bar{a}'_{i1}D_i)\ (a'_{i3}D_i)\|$ and the matrix $\|(a'_{i1}\bar{D}_i)\ (a'_{i3}\bar{D}_i)\|$. The web of quadrics which touch F_3 at P and $\overline{P}$ cuts F_3 in a web of sextic curves of genus two which is invariant under J_B. The locus of an additional point of contact with F_3 of quadrics of the web is the invariant nonic of J_B. It lies on the cubic surface K which has the equation

$$p\overline{H} - \bar{p}H = 0.$$

If the a_{ij}, b_{ij}, and c_{ij} are polynomials of order n in λ, the surfaces H, $\overline{H}$, L, and K are of orders $48n+13$, $48n+13$, $24n+6$, and $36n+9$ in λ. When λ is replaced by x_4/x_3, then as in §2 we can write the table of characteristics of the involutorial space transformation $I_{120n+51}$ as follows:

$$\begin{aligned}
l &\sim F_{24n+12}{:}\,l^{24n+7+t} + (O,\bar{O})^{24n+8} + C^{2}_{12n+6},\\
(O,\bar{O}) &\sim F_{96n+38}{:}\,l^{96n+26+5t} + (O,\bar{O})^{96n+31} + C^{4}_{12n+6},\\
C_{12n+6} &\sim F_{264n+112}{:}\,l^{264n+76+12t} + (O,\bar{O})^{264n+88} + C^{13}_{12n+6},\\
S_1 &\sim S_{120n+51}{:}\,l^{120n+34+6t} + (O,\bar{O})^{120n+40} + C^{6}_{12n+6},\\
&\quad K_{36n+18}{:}\,l^{36n+9+3t} + (O,\bar{O})^{36n+12} + C^{3}_{12n+6}.
\end{aligned}$$

The parasitic lines of the transformation are of three types. (a) There are the $24n+8$ trisecants of the C_{12n+6}. (b) The bisecants of C_{12n+6} through O or $\overline{O}$ are parasitic. In a plane λ through l there are fifteen bisecants through the six points of C_{12n+6} not on l, which determine fifteen points μ on l. Given a point μ on l there are $h'=36n+7$ bisecants of C_{12n+6} through it which determine h' planes λ. In the (λ, μ) correspondence there are $15+2h'$, ($2h'$, since $\mu^2=\lambda$), coincidences, or $72n+29$ positions of the points O, $\overline{O}$ such that bisecants of C_{12n+6} may be drawn through them. (c) In each of the $12n$ planes determined by l and the tangents to C_{12n+6} at its $12n$ intersections with l, the Bertini involution in the plane breaks down, and the line l is shed off. Hence there are $12n$ parasitic lines consecutive to l in these $12n$ planes. The total number of parasitic lines is $108n+37$.

To determine the number of parasitic conics and cubics we take the complete intersection of two $S_{120n+51}$ and an $S_{120n+51}$ with K_{36n+18}.

$$(120n+51)^2 = 120n+51+(120n+34)^2+36+36(12n+6)$$
$$+108n+73+8\zeta+27\eta,$$
$$(120n+51)(36n+18) = 36n+18+(36n+9)(120n+34)$$
$$+18+18(12n+6)+108n+37+4\zeta+9\eta.$$

The solution of these equations is $\zeta=144n+47$ conics and $\eta=84n+27$ cubics. The fundamental curves of the second species of the involutorial transformation $I_{120n+51}$ consist of $108n+37$ lines, $144n+47$ conics, and $84n+27$ cubics.

CORNELL UNIVERSITY

SOME THEOREMS ON DOUBLE LIMITS*

BY J. D. HILL†

1. *Introduction.* Let $f(x, y)$ be an arbitrary single-valued real function of the real variables x, y defined in the neighborhood of a point $Q(a, b)$, which for simplicity may be taken as $(0, 0)$. The following sufficient (and obviously necessary) condition for the existence of the double limit

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x, y) \tag{1}$$

has been established.

THEOREM 1 (Clarkson).‡ *If $f(x, y)$ has a unique limit as $P(x, y)$ approaches Q on every curve having a tangent at Q, the double limit* (1) *exists.*

The present note is concerned with similar theorems, and for definiteness we state at the outset that the assertion, "$f(P)$ has a limit λ as $P\to Q$ on a point set§ E having Q as a limit point (or $\lim_{P\to Q} f(P) = \lambda$, on E)" shall mean that for each $\epsilon > 0$ there exists a positive $\delta(\epsilon, E)$ such that $|f(P) - \lambda| < \epsilon$ for all points P of E satisfying the condition $0 < |x| + |y| < \delta$.

Theorem 1 naturally suggests a question which is answered by Lemma 1, for convenience in the statement of which we introduce the following definition.

DEFINITION OF PROPERTY L. A class $\{E\}$ of sets E, each having Q as a limit point, will be said to have Property L if and only if any set S whatsoever of points having Q as a limit

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* Presented to the Society, April 19, 1935.

† I gratefully acknowledge my indebtedness to Mr. Hugh J. Hamilton for suggesting Lemma 1, and to Mr. Nelson Dunford for Theorem 5.

‡ Clarkson, *A sufficient condition for the existence of a double limit*, this Bulletin, vol. 38 (1932), pp. 391–392. A theorem essentially the same has been proved by Verčenko and Kolmogoroff, *Über Unstetigkeitspunkte von Funktionen zweier Veränderlichen*, Comptes Rendus, Académie des Sciences, URSS, new series, vol. 1 (1934), pp. 105–107.

§ In particular, on a curve.

point has a subset S^* which is contained in some one of the sets E and has Q as a limit point.

LEMMA 1. *A necessary and sufficient condition that the relation* $\lim_{P\to Q} f(P)=\lambda$ *on every set* E *of a class* $\{E\}$ *shall imply the existence of* (1) *is that* $\{E\}$ *have Property L.*

This lemma, whose proof we leave to the reader, provides a criterion for determining whether or not an analog of Theorem 1 holds for other classes of curves or point sets.

2. *The Class of Curves* $\{\mathfrak{A}\}$. Let $\phi(s)\equiv\sum_{n=1}^{\infty}a_n s^n$, $\psi(s)\equiv\sum_{n=1}^{\infty}b_n s^n$ be any two real power series with positive radii of convergence (say) ρ_a, ρ_b, respectively, and let ρ be chosen so that $0<\rho<\min(\rho_a, \rho_b)$. Then the equations

$$x=\phi(s), \qquad y=\psi(s), \qquad (|s|\leqq\rho), \tag{2}$$

define a curve $\mathfrak{A}$ through Q. We denote by $\{\mathfrak{A}\}$ the class of all such curves.

THEOREM 2. *The existence of a unique limit for* $f(P)$ *as* $P\to Q$ *on every curve of* $\{\mathfrak{A}\}$ *does not imply the existence of* (1).

PROOF. Let us assume the contrary, which implies that $\{\mathfrak{A}\}$ has Property L. We choose S as the set of points on the curve $y=e^{-1/x^2}$ for $x>0$, and proceed to show that the definition of Property L is not satisfied. Suppose that there exists a curve $\mathfrak{A}^*$ of $\{\mathfrak{A}\}$ and an infinite subset S^* of S of points $(\xi_n, \eta_n)\to(0, 0)$, such that S^* lies on $\mathfrak{A}^*$. Then if (2) is the representation of $\mathfrak{A}^*$, there must exist at least one value of s, say σ_n, for which $\phi(\sigma_n)=\xi_n$, $\psi(\sigma_n)=\eta_n$, $(n=1, 2, 3, \cdots)$. Let λ be any limit point of the sequence $\{\sigma_n\}$, and let $\{s_n\}$ be a subsequence of $\{\sigma_n\}$ such that $s_n\to\lambda$ as $n\to\infty$. If $\{(x_n, y_n)\}$ is the corresponding subset of $\{(\xi_n, \eta_n)\}$, we have $0<x_n=\phi(s_n)\to 0$, and $0<y_n=\psi(s_n)\to 0$, whence by continuity $\phi(\lambda)=\psi(\lambda)=0$. Consequently, in view of the relation $|\lambda|\leqq\rho<\min(\rho_a, \rho_b)$, $\phi(s)$ and $\psi(s)$ have expansions of the form

$$\begin{aligned}\phi(s) &= \sum_{n=\mu}^{\infty}\alpha_n(s-\lambda)^n, \qquad (\mu\geqq 1, \alpha_\mu\neq 0),\\ \psi(s) &= \sum_{n=\nu}^{\infty}\beta_n(s-\lambda)^n, \qquad (\nu\geqq 1, \beta_\nu\neq 0),\end{aligned} \tag{3}$$

for $|s-\lambda|$ sufficiently small. Choose an integer m to satisfy the inequality $m\mu > \nu$, and consider the equation

$$\frac{\psi(s_n)}{[\phi(s_n)]^m} = \frac{e^{-1/x_n^2}}{x_n^m}, \qquad (n = 1, 2, 3, \cdots),$$

which is implied by $S^* \subset \mathfrak{A}^*$. Using (3) one sees that the left side increases without limit as $n \to \infty$, while the right side tends to zero. This contradiction completes the proof.

3. *The Class of Curves* $\{\mathfrak{B}_r\}$. Let r be a preassigned real number, or ∞, and denote by $\{\Gamma_r\}$ the class of all single-valued functions of $z(=s+it)$, each of which (i) is analytic in the extended plane except for a singularity at $z=r$, (ii) vanishes at $z=0$, and (iii) is real on the real axis. Then about $z=0$ each function in $\{\Gamma_r\}$ admits a power series expansion with real coefficients whose radius of convergence is $|r|$. Let $\{\Pi_r\}$ be the class of all such power series, and let $\{\mathfrak{B}_r\}$ be the class of all curves $\mathfrak{B}_r$ through Q each of which is defined parametrically by

$$x = \phi(s) \equiv \sum_{n=1}^{\infty} a_n s^n, \qquad y = \psi(s) \equiv \sum_{n=1}^{\infty} b_n s^n, \tag{4}$$

where the power series belong to the class $\{\Pi_r\}$.

THEOREM 3. *For each fixed* r, $(0 < |r| \leq \infty)$, *the existence of a unique limit for* $f(P)$ *as* $P \to Q$ *on every curve of* $\{\mathfrak{B}_r\}$ *implies the existence of* (1).†

This theorem is an immediate consequence of Lemma 1 and the following two lemmas, the first of which may be regarded as evident.

LEMMA 2. *Corresponding to each enumerable set* E *there exists a set* G *of points* (x_n, y_n) *with* $E \subset G$ *and* $|x_n|, |y_n| < n$, $(n = 1, 2, 3, \cdots)$.

LEMMA 3. *Corresponding to each enumerable set* E *there exists a curve* $\mathfrak{B}_r$ *of the class* $\{\mathfrak{B}_r\}$ *which passes through every point of* E.‡

PROOF. Setting

† It is worthy of note that, by Theorem 2, the existence of a unique limit for $f[\phi(s), \psi(s)]$ as s, $(|s| \leq r' < r)$, tends to zero for every curve of $\{\mathfrak{B}_r\}$ does *not* imply the existence of (1).

‡ It may well be that this lemma or something like it is known, but we have been unable to locate it in the literature.

$$g_m(w) \equiv 2(-1)^{m+1} \prod_{m \neq \nu=1}^{\infty} \left(1 - \frac{w^2}{\nu^2}\right) \equiv (-1)^{m+1} \frac{2m^2 \sin \pi w}{\pi w(m^2 - w^2)},$$

we have for $m = 1, 2, 3, \cdots$,

$$(5) \qquad \begin{aligned} &|g_m(w)| \leqq 2e^{k|w|^2}, \quad \text{where} \quad k = \sum_{\nu=1}^{\infty} 1/\nu^2, \\ &g_m(\pm m) = 1, \quad g_m(\pm n) = 0, \quad (m \neq n = 1, 2, 3, \cdots). \end{aligned}$$

We first assume r finite; let $\rho = |r|$ and μ be the greatest integer $\leqq 1/\rho$. Then there exists a σ satisfying the relation

$$(6) \qquad \rho m - 1 > \sigma > 0, \qquad (m = \mu + 1, \mu + 2, \cdots).$$

We define expressions c_n by the formula

$$(7) \qquad c_m = 1/[m^4(\rho m - 1)], \qquad (m = \mu + 1, \mu + 2, \cdots).$$

By Lemma 2 there exists a set G of points (ξ_n, η_n) with $G \supset E$ and $|\xi_n|, |\eta_n| < n$, $(n = 1, 2, 3, \cdots)$. Letting $m = \mu + n$, $x_m = \xi_n$, $y_m = \eta_n$, $(n = 1, 2, 3, \cdots)$, we have

$$(8) \qquad |x_m|, |y_m| < m - \mu \leqq m, \qquad (m = \mu + 1, \mu + 2, \cdots).$$

From (5), (6), (7), (8), we obtain

$$|c_m x_m g_m(w)|, \quad |c_m y_m g_m(w)| \leqq 2e^{k|w|^2}/\sigma m^3,$$

which shows that each of the infinite series

$$(9) \quad F_1(w) \equiv \sum_{m=\mu+1}^{\infty} c_m x_m g_m(w), \qquad F_2(w) \equiv \sum_{m=\mu+1}^{\infty} c_m y_m g_m(w)$$

converges uniformly in any finite region, and accordingly represents an entire function since $g_m(w)$ is entire. Moreover, since G may be assumed to include a point not on either axis, it is evident from the definitions of c_m and $g_m(w)$ that neither $F_1(w)$ nor $F_2(w)$ is a constant. Consequently

$$(10) \quad F_3(w) \equiv -w^4(rw + 1)F_1(w), \quad F_4(w) \equiv -w^4(rw + 1)F_2(w)$$

are entire functions with singularities at $w = \infty$. By means of the transformation

$$(11) \qquad w = 1/(z - r),$$

$F_3(w)$, $F_4(w)$ are transformed respectively into functions $\phi(z)$, $\psi(z)$ which belong to $\{\Gamma_r\}$ and thus determine a curve $\mathfrak{B}_r^*$ of the form (4). Finally [using (11), (10), (9), (7), and (5)] we obtain for $n=\mu+1, \mu+2, \cdots$

$$\phi(r - 1/n) = x_n, \ \psi(r - 1/n) = y_n, \text{ if } r > 0,$$
$$\phi(r + 1/n) = x_n, \ \psi(r + 1/n) = y_n, \text{ if } r < 0,$$

which proves that the curve $\mathfrak{B}_r^*$ passes through each point of G; E being a subset of G, the lemma is established for the case of r finite.

For $r=\infty$, the functions

$$\phi(z) \equiv z^4 \sum_{m=\mu+1}^{\infty} x_m g_m(z)/m^4, \qquad \psi(z) \equiv z^4 \sum_{m=\mu+1}^{\infty} y_m g_m(z)/m^4$$

which belong to $\{\Gamma_\infty\}$, lead to the same conclusion if z is assigned the values $n=\mu+1, \mu+2, \cdots$.

In passing it seems of interest to mention the following corollary.

COROLLARY. *There exists a curve $\mathfrak{B}_r$ of the class $\{\mathfrak{B}_r\}$ which passes through every point in the plane with rational coordinates.*

From Lemma 3 it is clear that the class $\{\mathfrak{B}_r\}$ has Property L; Theorem 2 then follows by Lemma 1.

4. *The Class of Curves* $\{\mathfrak{C}\}$. Let $F(x, y) \not\equiv 0$ be a real, single-valued function of the real variables x, y which is analytic in some neighborhood of Q and for which $F(0, 0) = 0$. Then $F(x, y) = 0$ defines a curve $\mathfrak{C}$ through Q. Excluding those curves for which Q is an isolated point, we denote by $\{\mathfrak{C}\}$ the class of all curves $\mathfrak{C}$ which remain. By employing a well known theorem of Weierstrass,† together with an analog of the Puiseux method for algebraic curves, one may readily verify that for each curve $\mathfrak{C}$ of $\{\mathfrak{C}\}$ there exists a neighborhood of Q in which all points of $\mathfrak{C}$ lie on a *finite* number of curves of class $\{\mathfrak{A}\}$. Combining this fact with the proof of Theorem 2 we obtain the following theorem.

THEOREM 4. *The existence of a unique limit for $f(P)$ as $P \to Q$ on every curve of $\{\mathfrak{C}\}$ does not imply the existence of* (1).

† Goursat-Hedrick-Dunkel, *Functions of a Complex Variable*, pp. 233 ff.

5. *The Class of Curves* $\{\mathfrak{D}\}$. Let $\{\mathfrak{D}\}$ denote the class of all curves $\mathfrak{D}$ representable parametrically as

$$x = x(s), \qquad y = y(s), \qquad (0 \leqq s \leqq 1),$$

where $x(s)$ and $y(s)$ have derivatives of all orders and $x(0) = y(0) = 0$.

THEOREM 5. *If* $f[x(s), y(s)]$ *has a unique limit as* s *tends to zero for every curve of* $\{\mathfrak{D}\}$, *the double limit* (1) *exists.*

PROOF. Let S be any set of points having the point Q as a limit point, and let S^* be a subset of points (x_n, y_n) tending to Q such that we have $|x_n|, |y_n| < e^{-1/n^2}$, $(n = 1, 2, 3, \cdots)$. If we set $I_1 \equiv (1/2 \leqq s \leqq 1)$, and $I_n \equiv [1/(n+1) \leqq s \leqq (2n+1)/(2n(n+1))]$, $(n = 2, 3, 4, \cdots)$, then the equations $x(0) = 0$, $x(s) = x_{n+1}$ for s in I_n, define a function with a closed domain which can be extended† to the whole interval $(0 \leqq s \leqq 1)$ in such a way that the extended function $x(s)$ has derivatives of all orders. The function $y(s)$ is defined similarly. The corresponding curve $\mathfrak{D}$ is such that the point $[x(s), y(s)]$ approaches Q through the set S^* as s tends to zero. This proves that $\{\mathfrak{D}\}$ has Property L, and establishes the theorem.

6. *The Class of Curves* $\{\mathfrak{E}\}$. Let $\{\mathfrak{E}\}$ be the class of all curves $\mathfrak{E}$ through Q, each of which has, with respect to a properly chosen system of rectangular coordinates ξ, η with origin at Q, an equation of the form $\eta = \phi(\xi)$, where $\phi(\xi)$ is a single-valued function with a continuous, non-negative, monotonic increasing first derivative in a certain neighborhood of $\xi = 0$ and $\phi'(0) = 0$. For a fixed system ξ, η denote by $x(\xi, \eta)$, $y(\xi, \eta)$ the coordinates of the point (ξ, η) in the original system x, y. Concerning the class of curves $\{\mathfrak{E}\}$ we have the following theorem which is an improvement over Theorem 1 to the extent that $\{\mathfrak{E}\}$ is a proper subclass of the class considered by Clarkson.

THEOREM 6. *If* $f[x(\xi, \phi(\xi)), y(\xi, \phi(\xi))]$ *has a unique limit as* ξ *tends to zero for every curve of* $\{\mathfrak{E}\}$, *the double limit* (1) *exists.*

PROOF. S being any set of points having Q as a limit point one readily sees by Clarkson's reasoning that axes ξ, η can be

† Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of this Society, vol. 36 (1934), pp. 63–89, Theorem 1.

so chosen that every closed sector lying in the first quadrant and having the ξ axis as one boundary will contain a subset of S having Q as a limit point. If S has a subset on the ξ axis with Q as a limit point, the curve $\eta=\phi(\xi)\equiv 0$ of class $\{\mathfrak{E}\}$ passes through a subset of S with the limit point Q, and the definition of Property L is satisfied. In the alternative case, we can, by the choice of axes, select a subset S^* of S of points (ξ_n, η_n) tending to Q, such that we have

$$0 < \xi_{n+1} < \xi_n/2, \qquad 0 < \eta_{n+1} < \eta_n/2,$$

$$\eta_n/\xi_n \to 0 \text{ as } n \to \infty, \qquad 0 < 2\eta_{n+1}/\xi_{n+1} < \eta_n/(2\xi_n).$$

From these relations it follows that

$$\frac{2\eta_{n+1}}{\xi_{n+1}} < \frac{\eta_n}{2\xi_n} < \frac{\eta_n - \eta_{n+1}}{\xi_n} < \frac{\eta_n - \eta_{n+1}}{\xi_n - \xi_{n+1}} < \frac{\eta_n}{\xi_n - \xi_{n+1}} < \frac{2\eta_n}{\xi_n};$$

hence $\sigma_n \equiv (\eta_n - \eta_{n+1})/(\xi_n - \xi_{n+1})$ tends monotonically to zero in the strict sense as $n \to \infty$. Consider the sequence of functions $\phi_n(\xi)$ defined as follows. Let $\phi_n(\xi) = \eta_{n+1} + \sigma_n(\xi - \xi_{n+1})$ on the interval $I_n \equiv (\xi_{n+1} \leqq \xi \leqq \xi_n)$ for n odd. For n even, let $\phi_n(\xi)$ be any function on I_n such that $\phi_n(\xi_{n+1}) = \eta_{n+1}$, $\phi_n(\xi_n) = \eta_n$, $\phi_n'(\xi_{n+1}+0) = \sigma_{n+1}$, $\phi_n'(\xi_n - 0) = \sigma_{n-1}$, and such that $\phi_n'(\xi)$ is continuous and increases monotonically from σ_{n+1} to σ_{n-1} as ξ increases from ξ_{n+1} to ξ_n. That such a function exists is clear from the fact that an arc of an ellipse† can be found whose equation satisfies these conditions.

In the interval $-\xi_1 < \xi < \xi_1$, let $\phi(\xi) = 0$ for $-\xi_1 < \xi \leqq 0$, and let $\phi(\xi) = \phi_n(\xi)$ on I_n, $(n = 1, 2, 3, \cdots)$. Then it is easily verified that the curve $\eta = \phi(\xi)$ is of class $\{\mathfrak{E}\}$, and by construction it passes through the set S^* as ξ tends to zero through positive values. This completes the proof that $\{\mathfrak{E}\}$ has Property L, and establishes Theorem 6.

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† Such an ellipse is given by the equation

$$[\eta - \eta_{n+2} - \sigma_{n+1}(\xi - \xi_{n+2})]\,[\eta - \eta_n - \sigma_{n-1}(\xi - \xi_n)] - k[\eta - \eta_{n+1} - \sigma_n(\xi - \xi_{n+1})]^2 = 0,$$

for each $k > (\sigma_{n-1} - \sigma_{n+1})^2/(4(\sigma_{n-1} - \sigma_n)(\sigma_n - \sigma_{n+1}))$.

THE RELATIVE CONNECTIVITIES OF SYMMETRIC PRODUCTS*

BY MOSES RICHARDSON

1. *Introduction.* The topology of the domain of discontinuity of a finite group of transformations operating on a complex, and, in particular, the topology of symmetric product complexes, has been studied by P. A. Smith† and the author.‡ Following a suggestion made by Morse,§ we obtain in this note explicit formulas for the so-called relative connectivities of the symmetric product of a complex in terms of its mod 2 Betti numbers, and we discuss an application of this result to the theory of critical chords. First, however, we derive a more general result of which the formulas for the relative connectivities of symmetric products is a special case. The methods used here follow closely those of S.

2. *Definitions and Preliminary Theorems.* For proofs or fuller discussion of statements made in this section, the reader is referred to S or R.

Let K be a simplicial n-complex.‖ Let T be a topological involution such that (a) T carries m-simplexes of K into m-simplexes of K; (b) if a simplex of K is invariant, it is pointwise invariant.

The invariant simplexes of K form a subcomplex K^0, and the non-invariant simplexes can be grouped in pairs so that each member of a pair is transformed into the other member by T. Thus the m-simplexes of K can be renamed E_m^i, $\overline{E}_m^i$, E_m^{0j}, where $\overline{E}_m^i = TE_m^i$, and E_m^{0j} is a simplex of K^0. If¶ $C = t_i E_m^i$ is a chain of

* Presented to the Society, February 23, 1935.

† P. A. Smith, *The topology of involutions*, Proceedings of the National Academy of Sciences, (1933), pp. 612–618. (Denoted hereafter by S.)

‡ M. Richardson, *On the homology characters of symmetric products*, Duke Mathematical Journal, vol. 1 (1935), pp. 50–69. (Denoted hereafter by R.)

§ M. Morse, *The Calculus of Variations in the Large*, Colloquium Publications of this Society, vol. 18, 1934, p. 191. (Denoted hereafter by M.)

‖ Our general topological terminology and notation is that of S. Lefschetz, *Topology*, Colloquium Publications of this Society, vol. 12, 1930.

¶ A repeated index indicates summation.

K, we define TC to be the chain $\overline{C}=t_i\overline{E}_m^i$. The involution T preserves bounding relations.†

We consider only mod 2 topology; all homologies and equations are understood to be homologies and congruences mod 2.

A chain X of K is called invariant if $X=\overline{X}$. In particular, if every simplex occurring in a chain with a non-zero coefficient belongs to K^0, we attach a zero to the chain-symbol, as X^0. If no simplex of K^0 occurs in a chain with a non-zero coefficient, we attach an asterisk to the chain-symbol, as X^*. Every invariant chain can be written in the form $X^*+\overline{X}^*+X^0$. If an invariant cycle Γ is the boundary of an invariant chain, we write $\Gamma\cong 0$. These *special* homologies obey the same formal rules as ordinary homologies.

We choose a base for homology of type Γ^i, $\overline{\Gamma}^i$, D^j for each dimension.‡ We consider only the case in which (A) $D_m^j+\overline{D}_m^j\cong 0$ for every $m>0$ and every j. In this case we can and do replace the D_m^i in the base by invariant cycles§ ${}^i\Delta_m$.

We now construct the sequences

$$(1)\qquad {}^i\Delta_m^m,\ {}^i\Delta_{m-1}^m,\ \cdots,\ {}^i\Delta_r^m,\qquad (r=r(m,i)\geqq -1),$$

where ${}^i\Delta_q^m={}^iX_q^m+{}^i\overline{X}_q^m$, and $F({}^iX_q^m)={}^i\Delta_{q-1}^m$, (for $q=m, m-1, \cdots, r+1$), and ${}^i\Delta_r^m={}^iX_r^m+{}^i\overline{X}_r^m+{}^iX_r^{0m}$, (for $r\geqq 0$), where the ${}^iX_r^{0m}$ are cycles.‖ We consider only the case where (B) the cycles ${}^iX_r^{0m}$ are independent with respect to homologies on K^0. We shall need the following lemmas.¶

(2) *If* $C+\overline{C}+C^0\cong 0$, *then* $C^0\sim 0$ *on* K^0.

(3) *The cycles* $\Gamma_q^i+\overline{\Gamma}_q^i$, ${}^i\Delta_q^m$, $(q>0)$, *are independent with respect to* $\cong$.

(4) *If* (A) *and* (B) *hold, every cycle of the form* $C_q+\overline{C}_q$, $(q>0)$, *is* $\cong$ *to a linear combination of cycles* $\Gamma_q^i+\overline{\Gamma}_q^i$, ${}^i\Delta_q^m$.

With the simplexes E_m^i and $\overline{E}_m^i$ we associate† a simplex e_m^i, and we write $\wedge E_m^i=\wedge\overline{E}_m^i=e_m^i$. If $C=t_iE_m^i$, we define $\wedge C$ to be the chain $c=t_ie_m^i$. The totality of simplexes e_m^i constitutes

† R, §1.
‡ S, §1.
§ S, p. 614.
‖ S, p. 614.
¶ S, pp. 613–615.
† The material in this paragraph is fully discussed in R, §2.

an n-complex $k=\wedge K$, say. In particular, the simplexes e_m^{0i} $=\wedge E_m^{0i}$ constitute a subcomplex $k^0=\wedge K^0$, say. If $e=\wedge E$ is a simplex of k, we write $\wedge' e=E+\overline{E}$. If $c=t_i e_m^i$, we define $\wedge' c$ to be the chain $t_i \wedge' e_m^i$. Both $\wedge$ and $\wedge'$ preserve bounding relations. We shall use large or small letters for chains of K or k, respectively. In particular, a symbol like x^0 will denote a chain of k^0, and a symbol like x^* will denote a chain in which no cell of k^0 occurs with a non-zero coefficient.

3. *The Topology of k mod k^0.* We shall now determine the Betti numbers $R_q(k; k^0, 2)$. A chain whose boundary is a chain of k^0, that is, a cycle mod k^0, shall be called a relative cycle.

(5) *If $c+x^0 \to 0$, then c is a relative cycle.*

PROOF. Let $F(c)=y^*+y^0$ and $F(x^0)=z^0$. Since $F(c+x^0)$ $=y^*+y^0+z^0=0$, we have $y^*=0$. Hence $c \to 0$ mod k^0.

(6) *If $c+x^0 \sim 0$, then $c \sim 0$ mod k^0.*

PROOF. There exists a chain d such that $d \to c+x^0$. Thus $d \to c$ mod k^0.

(7) *If γ is a relative cycle, then $\wedge'\gamma$ is a cycle.*

PROOF. Since $\gamma \to x^0$, we have $\wedge'\gamma \to \wedge' x^0 = X^0+\overline{X}^0=2X^0=0$.

(8) *If $\gamma \sim 0$ mod k^0, then $\wedge'\gamma \cong 0$.*

PROOF. Since there exists a chain c such that $c \to \gamma+x^0$, we have $\wedge' c \to \wedge'\gamma+\wedge' x^0$. But $\wedge' x^0=0$. It is obvious that $\wedge' c$ and $\wedge'\gamma$ are invariant.

(9) *If c is a relative cycle, we can write $\wedge' c$ in the form $C+\overline{C}$ where $\wedge C=c$.*

PROOF. Let $c=t_i e_m^i+u_i e_m^{0i}$. We have only to let $C=t_i E_m^i$ $+u_i E_m^{0i}$.

(10) *If $C+\overline{C} \cong 0$, then $\wedge C$ is a relative cycle and $\wedge C \sim 0$ mod k^0.*

PROOF. By hypothesis, $H+\overline{H} \to C+\overline{C}$. Let $F(H)=C+X$. Then $X+\overline{X}=0$. Hence $X=X^*+\overline{X}^*+X^0$. Therefore,

$$\wedge H \to \wedge C + \wedge X = \wedge C + 2\wedge X^* + \wedge X^0 = \wedge C + \wedge X^0.$$

Thus $\wedge C+\wedge X^0 \to 0$, and by (5), $\wedge C$ is a relative cycle. Since $\wedge H \to \wedge C+\wedge X^0 \sim 0$, we have $\wedge C \sim 0$ mod k^0, by (6).

(11) *For $q \geqq r(m, i)+1$, $\wedge({}^iX_q^m)$ is a relative cycle, say ${}^i\xi_q^m$, and $\wedge'({}^i\xi_q^m) = {}^i\Delta_q^m$.*

PROOF. If $q \geqq r(m, i)+2$, then, since ${}^iX_q^m \to {}^i\Delta_{q-1}^m$, we have

$$\wedge({}^iX_q^m) \to \wedge({}^i\Delta_{q-1}^m) = \wedge({}^iX_{q-1}^m + {}^i\overline{X}_{q-1}^m) = 0.$$

Therefore, $\wedge({}^iX_q^m) = {}^i\xi_q^m$ is an absolute cycle. If $q = r(m, i)+1$, we have

$$\begin{aligned}\wedge({}^iX_q^m) \to \wedge({}^i\Delta_{q-1}^m) &= \wedge({}^iX_r^m + {}^i\overline{X}_r^m + {}^iX_r^{0m}) \\ &= \wedge({}^iX_r^{0m}) = 0 \bmod k^0.\end{aligned}$$

Thus, in this case, ${}^i\xi_q^m = \wedge({}^iX_q^m)$ is a relative cycle. In either case we have $\wedge'({}^i\xi_q^m) = {}^iX_q^m + {}^i\overline{X}_q^m = {}^i\Delta_q^m$.

Let $\gamma_q^i = \wedge\Gamma_q^i$.

(12) *The relative cycles γ_q^i, ${}^i\xi_q^m$, $(q \geqq r(m, i)+1 > 0)$, are independent with respect to homology* mod k^0.

PROOF. Suppose there were a non-trivial homology

$$x_{im}{}^i\xi_q^m + y_i\gamma_q^i \sim 0 \bmod k^0.$$

By (8) we have $\wedge'(x_{im}{}^i\xi_q^m + y_i\gamma_q^i) \cong 0$. Thus, by (11),

$$x_{im}{}^i\Delta_q^m + y_i(\Gamma_q^i + \overline{\Gamma}_q^i) \cong 0,$$

contradicting (3).

(13) *Every relative q-cycle of k is homologous* mod k^0 *to a linear combination of the γ_q^i and ${}^i\xi_q^m$, $(q \geqq r(m, i)+1 > 0)$.*

PROOF. Let γ be an arbitrary relative q-cycle of k. Let $\wedge'\gamma = \Gamma + \overline{\Gamma}$, where $\wedge\Gamma = \gamma$, by (9). By (7), $\wedge'\gamma$ is a cycle. Therefore, by (4),

$$\Gamma + \overline{\Gamma} \cong x_i(\Gamma_q^i + \overline{\Gamma}_q^i) + y_{im}{}^i\Delta_q^m. \tag{14}$$

Now we shall show that $y_{im} = 0$ whenever $r(m, i) = q$. For, if some $y_{im} \neq 0$, then (14) would be of the form

$$Y + \overline{Y} + z_{im}({}^iX_q^m + {}^i\overline{X}_q^m + {}^iX_q^{0m}) \cong 0,$$

where some $z_{im} \neq 0$. This implies that $z_{im}{}^iX_q^{0m} \sim 0$ on K^0, by (2). But this contradicts (B). Thus, (14) has the form

$$\Gamma + \overline{\Gamma} + x_i(\Gamma_q^i + \overline{\Gamma}_q^i) + y_{im}({}^iX_q^m + {}^i\overline{X}_q^m) \cong 0,$$

where $q \geqq r(m, i)+1$. Let $C=\Gamma+x_i\Gamma_q^i+y_{im}{}^iX_q^m$. Then $C+\bar{C}\cong 0$, and, by (10), we have $\wedge C\sim 0 \bmod k^0$, or

$$\gamma + x_i\gamma_q^i + y_{im}{}^i\xi_q^m \sim 0 \bmod k^0.$$

This proves the theorem.

By (12) and (13), the relative cycles ${}^i\xi_q^m$, γ_q^i, $(q \geqq r(m, i)+1 > 0)$, constitute a base for relative q-cycles of k with respect to homology mod k^0. Let R_q^Γ be the number of cycles Γ_q^i, and let Q_q be the number of q-cycles ${}^i\Delta_q^m$ satisfying the relation $r(m, i)+1 \leqq q$. We have proved the following theorem.

THEOREM 1. *If the hypotheses* (a), (b), (A), *and* (B) *are fulfilled, then* $R_q(k; k^0, 2)=R_q^\Gamma+Q_q, (q>0)$.

4. *Symmetric Products.* Let $K_{2n}=K_n\times K_n$ be the complex K of the preceding sections. Let T be the involution which interchanges the points $P\times Q$ and $Q\times P$ of K_{2n}. A simplicial subdivision of K_{2n} satisfying (a) and (b) of §2 can be found.† Of course, $\wedge K_{2n}=k_{2n}$ is the 2-fold symmetric product of K_n. We can choose bases for homology on K_{2n} of the Γ, $\bar{\Gamma}$, ${}^i\Delta$ type here required.‡ The cycles ${}^i\Delta_q$ occur only in even dimensions. It has been shown that the sequences (1) can be constructed so that $r(2h, i)=h$ for all i, and so that (A) and (B) are fulfilled.§ Therefore we may apply Theorem 1.

Now let $R_{2s}^\Delta=R_s(K_n, 2)$ for $s \leqq n$ and $R_{2s}^\Delta=0$ for $s>n$. Then it is easily seen that $Q_1=0$ and

$$Q_q = R_{2t}^\Delta + R_{2(t+1)}^\Delta + \cdots + R_{2(q-1)}^\Delta, \qquad (q > 1),$$

where $t=[(q+1)/2]$, since the lowest dimension $2m$ to yield cycles ${}^i\Delta_q{}^{2m}$ is either $2m=q$ or $2m=q+1$. Thus by Theorem 1, we have the following result.

THEOREM 2. *For the symmetric product* k_{2n} *of* K_n *we have*

$$R_1(k_{2n}; k_n^0, 2) = R_1^\Gamma$$
$$R_q(k_{2n}; k_n^0, 2) = R_q^\Gamma + R_{2t}^\Delta + R_{2(t+1)}^\Delta + \cdots + R_{2(q-1)}^\Delta, \quad (q > 1),$$

where $t=[(q+1)/2]$, *and where*

† R, §5.
‡ R, p. 57.
§ R, pp. 64–65.

$$R_q^{\Gamma} = \frac{1}{2}[R_q(K_{2n}, 2) - R_q^{\Delta}]$$

if q is even, and

$$R_q^{\Gamma} = \frac{1}{2} R_q(K_{2n}, 2)$$

if q is odd.

Of course, if K_n is connected, so is k_{2n}; hence $R_0(k_{2n}; k_n^0, 2) = 0$ in this case. The numbers $R_q(k_{2n}; k_n^0, 2)$ have been called relative connectivities by Morse,† who proved that they are finite.‡ This result is of course implied by our formulas.

EXAMPLE 1. Let K_n be an n-sphere. Then $R_n^{\Gamma} = R_{2n}^{\Delta} = 1$, while all the other R^{Γ}'s and R^{Δ}'s, are zero. From our formulas, we obtain for the relative connectivities of k_{2n},

$$\begin{aligned}
R_0 &= R_1 = \cdots = R_{n-1} = 0, \\
R_n &= R_n^{\Gamma} = 1, \\
R_{n+1} &= R_{2(q-1)}^{\Delta} = R_{2n}^{\Delta} = 1, \\
R_{n+2} &= R_{2(q-2)}^{\Delta} = R_{2n}^{\Delta} = 1, \\
&\cdots\cdots\cdots\cdots \\
R_{2n} &= R_{2t}^{\Delta} = R_{2n}^{\Delta} = 1.
\end{aligned}$$

The values of the relative connectivities for this example were worked out by Morse§ by special methods involving the critical chords of an n-ellipsoid.

EXAMPLE 2. Let K_n be an orientable surface of genus p. Then the relative connectivities of the symmetric product k_{2n} are $R_0 = 0$, $R_1 = 2p$, $R_2 = 2p^2 + p + 1$, $R_3 = 2p + 1$, $R_4 = 1$.

5. *Application to the Theory of Critical Chords.*‖ The chief results concerning critical chords are as follows. Let R be a regular, analytic Riemannian n-manifold lying in a euclidean $(n+1)$-space, such that R is homeomorphic to a simplicial n-complex K_n. Then the symmetric product of R is evidently homeo-

† M, p. 182.

‡ M, pp. 182–183.

§ M, Theorem 11.3, p. 191.

‖ For definitions and proofs required in this section see M, pp. 181–191.

morphic to the symmetric product k_{2n} of K_n. Let $R_0, R_1, \cdots, R_{2n}$ be the relative connectivities of k_{2n}. Then the sums M_i of the type numbers of the critical sets of chords of R and the numbers R_i satisfy the relations

$$M_0 \geqq R_0, M_0 - M_1 \leqq R_0 - R_1, M_0 - M_1 + M_2 \geqq R_0 - R_1 + R_2,$$

. . . . . .

$$M_0 - M_1 + \cdots + (-1)^{2n} M_{2n} = R_0 - R_1 + \cdots + (-1)^{2n} R_{2n}.$$ †

A simple corollary of this theorem is this: *If the critical chords of R are all non-degenerate, there exist at least R_i such chords of index‡ i.*

Our Theorem 2 enables us to obtain the values of the relative connectivities R_i of k_{2n} when the mod 2 Betti numbers of R are known. Thus the above theorem and its corollary can be used to obtain numerical information concerning the critical chords of any R whose mod 2 Betti numbers are known. This makes available a wide class of examples. For instance, the corollary of M, p. 191 follows at once from the above corollary and our Example 1, §4.

As a further example, let R be any regular, analytic image of an orientable surface of genus p. Then, from Example 2, §4, and the above corollary, we obtain the result that, if the extremal chords of R are all non-degenerate, then, among these extremal chords there must be $2p^2+5p+3$ extremal chords of the following description: $2p$ extremal chords of index 1, $2p^2+p+1$ extremal chords of index 2, $2p+1$ extremal chords of index 3, and 1 extremal chord of index 4. In the degenerate case, the same result holds provided each critical set of chords is counted according to its type numbers.

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† M, Theorem 11.1, p. 185.

‡ M, p. 185.

THE BLOCH CONSTANT $\mathfrak{A}$ FOR A SCHLICHT FUNCTION*

BY R. M. ROBINSON

The following theorem† is due to Bloch.

There is an absolute positive constant P with the following property. Let $f(x)$ be regular for $|x|<1$, $f'(0)=1$. Then $y=f(x)$ maps the circle $|x|<1$ on a region (in a Riemann surface over the y plane) containing a circle of radius P in a single sheet.

(Without the condition $f'(0)=1$ there is a circle of radius $P|f'(0)|$.)

Landau‡ defines three absolute constants connected with this theorem. $\mathfrak{B}$ is the upper bound of the P which satisfy the theorem as stated. $\mathfrak{L}$ is the upper bound of the P if we require only that there be a circle of radius P in the y plane each point of which is covered by some sheet of the map. $\mathfrak{A}$ is the corresponding bound if $f(x)$ is given as schlicht (that is, $f(x_1)\neq f(x_2)$ for $x_1\neq x_2$, so that the map has only one sheet).

We have clearly $\mathfrak{B}\leqq\mathfrak{L}\leqq\mathfrak{A}$. The chief object of the paper by Landau is to give as close lower and upper bounds as possible for these three constants. He proves $0.39<\mathfrak{B}<0.56$, $0.43<\mathfrak{L}<0.56$, and $\mathfrak{A}>0.56$ (so that $\mathfrak{L}<\mathfrak{A}$). However, as an upper bound for $\mathfrak{A}$ he obtains no new result, but mentions a result of Szegö $\mathfrak{A}\leqq\pi/4$, as the best result which he knows. This follows from consideration of the function

$$y=\frac{1}{2}\log\frac{1+x}{1-x}=x+\cdots,$$

which maps the circle $|x|<1$ on the strip $|\Im y|<\pi/4$.

A better bound than $\pi/4$ can be obtained by mapping $|x|<1$ on as simple a region as a circle slit along a radius from the center to the circumference (Theorem 1). Still better bounds may be

* Presented to the Society, October 27, 1934.

† For a proof and further references, see a paper by Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634.

‡ Loc. cit.

obtained by using several slits extending along radii part way to the center (Theorems 2, 3, 4). Using this method I shall show $\mathfrak{A} < 0.66$.

LEMMA 1. *If*

$$k(x) = \frac{x}{(1-x)^2}, \qquad (|x| < 1),$$

and K is the inverse function, and we put

$$p = p(r) = \frac{4r}{(1+r)^2}, \qquad (0 < r < 1),$$

then $z = K(pk(x))$ maps the circle $|x| < 1$ on the circle $|z| < 1$ slit from -1 to $-r$.

PROOF. This is an immediate consequence of the fact that $k(x)$ maps the circle $|x| < 1$ on a plane cut from $-1/4$ to $-\infty$.*

THEOREM 1. *An upper bound for $\mathfrak{A}$ is given by the formula*

$$\mathfrak{A} \leqq \frac{3 + 2(2)^{1/2}}{8} \; (< 0.729).$$

PROOF. If we combine the transformations

$$z = K(pk(x)) = px + \cdots, \quad y = \frac{z+r}{rz+1} = r + (1-r^2)z + \cdots,$$

we have a transformation $y = f(x)$ which maps the circle $|x| < 1$ on the circle $|y| < 1$ slit from -1 to 0. We have

$$f'(0) = p(1-r^2) = \frac{4r(1-r)}{1+r}.$$

We choose r so that $f'(0)$ is a maximum;

$$r = (2)^{1/2} - 1, \qquad f'(0) = 4((2)^{1/2} - 1)^2.$$

Since the map contains no circle of radius greater than $1/2$, we have

$$\mathfrak{A} f'(0) \leqq \frac{1}{2}, \qquad \mathfrak{A} \leqq \frac{3 + 2(2)^{1/2}}{8}.$$

* See Landau's book, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 1929, pp. 112–113.

LEMMA 2. *If a function $F(x)$, regular for $|x|<1$ and such that $F(0)=0$, maps the circle $|x|<1$ on a plane region R, then $F(x^n)$ maps $|x|<1$ on n copies of R connected by a branch point at the origin, so that $[F(x^n)]^{1/n}$ maps $|x|<1$ on a plane region which is n-fold symmetric with respect to the origin.*

This known result is so evident as to require no proof.

THEOREM 2. *An upper bound for $\mathfrak{A}$ is given by the formula*

$$\mathfrak{A} \leqq \frac{3}{8}(6)^{1/3}(< 0.682).$$

PROOF. In Lemma 2, put $F(x)=K(pk(x))$ and $n=3$. This leads to the result that $y=f(x)=[K(pk(x^3))]^{1/3}$ maps the circle $|x|<1$ on the circle $|y|<1$ with three cuts along equally spaced radii to within a distance $(r)^{1/3}$ of the origin. For this function $f'(0)=(p)^{1/3}$. If we choose $r=1/8$, then the three cuts extend half way to the origin (Fig. 1). The largest circles that can be

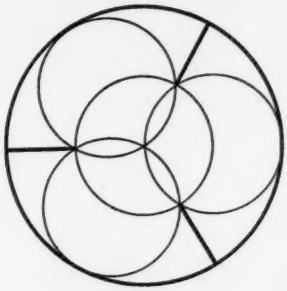

FIG. 1.

drawn in the map are of radius $1/2$, there being four of this size. Hence

$$\mathfrak{A}(p)^{1/3} \leqq \frac{1}{2}, \qquad \mathfrak{A} \leqq \frac{3}{8}(6)^{1/3}.$$

LEMMA 3. *If*

$$p_1 = p(r_1), \qquad p_2 = p(r_2), \qquad (0 < r_1 < 1,\ 0 < r_2 < 1),$$

and p is determined from the relation

$$\frac{1}{pp_1} = \frac{1}{p_1} + \frac{1}{p_2} - 1,$$

then the function $F(x)=K(p_1k(-K(pk(x))))$ *maps* $|x|<1$ *on a unit circle cut from* -1 *to* $-r_1$ *and from* 1 *to* r_2. *For this function*

$$\frac{1}{|F'(0)|}=\frac{1}{4}\left(r_1+\frac{1}{r_1}+r_2+\frac{1}{r_2}\right).$$

PROOF. $K(pk(x))$ maps $|x|<1$ on a unit circle cut from -1 to $-r$. For $-K(pk(x))$ the cut is from 1 to r. Proceeding to $F(x)$ inserts a second cut from -1 to $-r_1$ as required, but distorts the first cut. The symmetry of the figure with respect to the real axis shows that the cut will still lie along the real axis. Its end point r is taken to $K(p_1k(r))$, which is equal to r_2, as required, if $p_1k(r)=k(r_2)$. Since

$$\frac{1}{k(r)}=4\left(\frac{1}{p(r)}-1\right),$$

this becomes

$$\frac{1}{p_1}\,4\left(\frac{1}{p}-1\right)=4\left(\frac{1}{p_2}-1\right),\qquad \frac{1}{pp_1}=\frac{1}{p_1}+\frac{1}{p_2}-1.$$

Since $|F'(0)|=pp_1$, this gives the last statement of the lemma if we put in the values of p_1 and p_2.

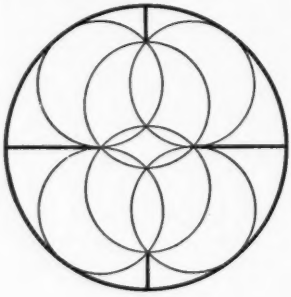

FIG. 2.

THEOREM 3. *An upper bound for* $\mathfrak{A}$ *is given by the formula*

$$\mathfrak{A}^2\leqq\frac{(1+v^2)^4(1+v-v^2)}{16v^2(1+v)^3(1-v)}\quad \textit{for}\quad (2)^{1/2}-1\leqq v<1.$$

Putting $v=0.55$, *we find* $\mathfrak{A}<0.666$.

PROOF. We shall map the circle $|x|<1$ on a unit circle with cuts from 1 to u, -1 to $-u$, i to vi, $-i$ to $-vi$ (Fig. 2). The quantities u, v are to be so related that each of the four circles tangent to the real axis at $\pm u$, and tangent to the unit circle, shall pass through one of the points $\pm vi$. Let the radius of these circles be ρ. Then

$$u^2+\rho^2=(1-\rho)^2, \qquad u^2+(v-\rho)^2=\rho^2,$$

from which we find

$$u^2=\frac{v(1-v)}{1+v}, \qquad \rho=\frac{1+v^2}{2(1+v)}.$$

If $v\geqq(2)^{1/2}-1$, the maximum circles in the map are six of radius ρ, the four mentioned, and two through $\pm u$ and one of the points $\pm vi$. (For $v=(2)^{1/2}-1$, $u=(2)^{1/2}-1$ and four of the circles are tangent to both axes, the other two coinciding at the center.) Hence if $f(x)$ is the mapping function, $\mathfrak{A}|f'(0)|\leqq\rho$. To accomplish the mapping, we first map $|x|<1$ on a unit circle with cuts from -1 to $-v^2$ and from 1 to u^2 by the function $F(x)$ of Lemma 3 (putting $r_1=v^2$, $r_2=u^2$). For this function

$$\frac{1}{|F'(0)|}=\frac{1}{4}\left(v^2+\frac{1}{v^2}+u^2+\frac{1}{u^2}\right)=\frac{(1+v^2)^2(1+v-v^2)}{4v^2(1+v)(1-v)}.$$

We then put $f(x)=[F(x^2)]^{1/2}$ for the final mapping function. Since $|f'(0)|^2=|F'(0)|$, we have

$$\mathfrak{A}^2|F'(0)|\leqq\rho^2,$$

which gives the theorem.

THEOREM 4. *An upper bound for* $\mathfrak{A}$ *is given by the formula*

$$\mathfrak{A}^3\leqq\frac{9}{4}\left(\frac{1}{a^6}+\frac{1}{8}\right),$$

where $a=1+(b^2+2b+2)^{1/2}$, $b=[2(3)^{1/2}-3]^{1/2}$. *Hence* $\mathfrak{A}<0.658$.

PROOF. We shall map on a region constructed as follows. Start with a circle of radius 2ρ about the origin, with three cuts half way to the center (the dotted circle in Fig. 3). It contains four circles of radius ρ, as in Fig. 1. The three cuts are extended outward, and three more cuts are drawn outward from the cir-

cumference. Six circles of radius ρ are then pushed as far as possible toward the center in the six sections. Consider, for example, the circle in the upper left section. We may slide it along the real axis toward the origin until it strikes the point $-\rho+(3)^{1/2}\rho i$, the end of one of the cuts. Its center then reaches the position

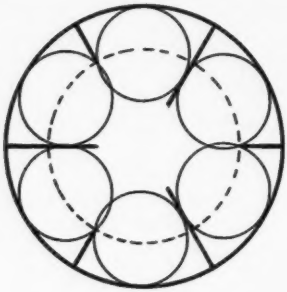

FIG. 3.

$-(b+1)\rho+\rho i$, at a distance $\rho(b^2+2b+2)^{1/2}$ from the origin. A circle of radius $a\rho$ will then include the six circles. We choose $\rho=1/a$, so that the latter circle is the unit circle. We thus are to map on a unit circle with three cuts extending to within a distance $1/a$ and three extending to within a distance $2/a$ from the origin. This is accomplished by taking the $F(x)$ of Lemma 3 with $r_1=1/a^3$, $r_2=8/a^3$ and then putting $y=f(x)=[F(x^3)]^{1/3}$. Now

$$\frac{1}{|F'(0)|}=\frac{1}{4}\left(r_1+\frac{1}{r_1}+r_2+\frac{1}{r_2}\right)=\frac{9}{4}\left(\frac{1}{a^3}+\frac{a^3}{8}\right).$$

Since $|f'(0)|^3=|F'(0)|$ and the map contains no circle of radius greater than $\rho=1/a$, we have

$$\mathfrak{A}|f'(0)|\leqq\frac{1}{a},\qquad \mathfrak{A}^2|F'(0)|\leqq\frac{1}{a^2},$$

which gives the theorem.

THE UNIVERSITY OF CALIFORNIA AT BERKELEY

ON THE SUMMABILITY OF A CERTAIN CLASS OF SERIES OF JACOBI POLYNOMIALS*

BY A. P. COWGILL

1. *Introduction.* The result obtained in this paper is as follows:

The series

$$\sum_{n=1}^{\infty} n^i \frac{(p+1)(p+3)\cdots(p+2n-1)}{2^n n!} X_n^{(p-1)/2}(x),$$

where $X_n^{(p-1)/2}(x)$ *(hereafter indicated simply by* X_n*) is a symmetric Jacobi polynomial,*† $p>-1$, *and* i *a positive integer, is summable* (C, k), $k>i-1/2$, *for the range* $-1<x<1$.

The proof is limited to symmetric Jacobi polynomials because of the necessity of having the recursion formula‡ of first degree in n. Unless explicitly stated otherwise, x is confined to the range $-1<x<1$, and $p>-1$, throughout this paper.

In the proof the sum of the n first terms of the given series is transformed, following the method employed by Brenke for Hölder summability of certain series of Legendre polynomials,§ by the recursion formula for Jacobi polynomials into a new sum of n terms, plus four additional terms. Then convergence factors for summability $(C, i-1)$ are applied, followed by those of summability (C, j), $j>1/2$, necessary to evaluate the additional

* Presented to the Society, November 30, 1934. This paper is a portion of a thesis presented to the Faculty of the Graduate College, The University of Nebraska.

† This is denoted by $F(-n, n+p, (p+1)/2, (1-x)/2)$ in the notation of Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–60, 377–416; p. 22. It is $G_n(p, (p+1)/2, (1-x)/2)$ in the notation of R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, vol. 1, p. 74. It is $X_n^{((p-1)/2,\ (p-1)/2)}(x)$ in the notation of G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, pp. 93–94, where the orthogonality property is expressed by means of the equality $\int_{-1}^{1}(1-x)^{(p-1)/2}(1+x)^{(p-1)/2}X_n^{((p-1)/2,(p-1)/2)}X_m^{((p-1)/2,(p-1)/2)}dx = 0 (m \neq n;\ m, n = 0, 1, \cdots)$.

‡ Darboux, loc. cit., p. 378.

§ W. C. Brenke, *On the summability and generalized sum of a series of Legendre polynomials*, this Bulletin, vol. 39 (1933), pp. 821–824.

terms involving n, making the equivalent of summability (C, k), $k>i-1/2$. The first application of summability causes the highest ordered part of the sum of the two additional terms involving n to take the form of a series, which is equal to the Cauchy product of two series, one of which is summable to a finite value and the other to the value zero.

Certain well known theorems of summability are used.*

THEOREM 1. *If two series are respectively summable* (C, α) *and* (C, β), *their Cauchy product is summable* $(C, \alpha+\beta+1)$ *to the product of the sums of the series.*

THEOREM 2. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* (C, δ) *to the value* s, *this implies that* $\lim_{x\to 1-0} \sum_{n=0}^{\infty} u_n x^n = s$.

THEOREM 3. *A series that is bounded* (C, α), $\alpha \geqq -1$, *is summable* $(C, \delta>\alpha)$ *with the sum* s, *if* $\lim_{x\to 1-0} \sum_{n=0}^{\infty} u_n x^n = s$.

THEOREM 4. *The summability* (C, δ) *of* $\sum u_n$ *implies* $u_n = o(n^\delta)$ *and* $s_n = o(n^\delta)$.

THEOREM 5. *The existence of* $\lim_{n\to\infty} s_n^{(\delta+\gamma)} = s$, *where* $s_n^{(\delta+\gamma)}$ *denotes the Cesàro nth partial mean of order* $\delta+\gamma$, *implies that of the double mean* $S_n^{(\delta,\gamma)}$ *of orders* δ *and* γ:

$$\lim_{n\to\infty} S_n^{(\delta,\gamma)} = \lim_{n\to\infty} S_n^{(\gamma,\delta)} = \lim_{n\to\infty} s_n^{(\delta+\gamma)},$$

and vice versa, provided δ, γ, $\delta+\gamma > -1$.

2. *The Polynomial* X_n. The recursion formula is†

$$xX_n = \frac{n+p}{2n+p} X_{n+1} + \frac{n}{2n+p} X_{n-1}. \tag{1}$$

The generating function,‡ when

$$a_n = \frac{(p+1)(p+3)\cdots(p+2n-1)}{2^n n!} = \frac{\Gamma((p+1)/2+n)}{\Gamma((p+1)/2)\Gamma(n+1)}$$

$$= O(n^{(p-1)/2}),$$

is

* E. Kogbetliantz, Mémorial des Sciences Mathématiques, vol. 51, pp. 19–20, 29, 37, 30–31, and 23, respectively.

† Darboux, loc. cit., p. 378.

‡ Darboux, loc. cit., p. 23.

$$(2)\qquad \frac{((1-x^2)/4)^{(1-p)/2}[2tx-2+2(1-2tx+t^2)^{1/2}]^{(p-1)/2}}{(2t)^{p-1}(1-2tx+t^2)^{1/2}} = \sum_{r=0}^{\infty} a_r X_r t^r, \qquad (0<t<1).$$

I find by the method used for Legendre polynomials by Byerly* that X_r in the following formula

$$(3)\qquad \sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r t^r = \frac{1}{(1-2tx+t^2)^{p/2}}, \quad (0<t<1),$$

satisfies the Jacobi recursion formula (1). We have also

$$X_n = O(n^{-p/2}), \qquad (-1<x<1).\dagger$$
$$X_n(1) = 1.\ddagger$$

3. *Transformation of the Series.* Multiplying (1) by c_n and summing from 1 to n, we get

$$(4)\qquad \begin{aligned} x\sum_{r=1}^{n} c_r X_r &= \sum_{r=1}^{n} \frac{r+p}{2r+p} c_r X_{r+1} + \sum_{r=1}^{n} \frac{r}{2r+p} c_r X_{r-1} \\ &= \sum_{r=1}^{n} \left[\frac{r-1+p}{2r-2+p} c_{r-1} + \frac{r+1}{2r+2+p} c_{r+1} \right] X_r \\ &\quad - c_0 X_1 + \frac{n+p}{2n+p} c_n X_{n+1} + \frac{1}{2+p} c_1 X_0 \\ &\quad - \frac{n+1}{2n+2+p} c_{n+1} X_n. \end{aligned}$$

If $c_n = n^i a_n$, the coefficient of X_r in (4) takes the form

$$U_r \equiv [r^i + b_1(p) r^{i-2} + b_2(p) r^{i-3} + \cdots] a_r,$$

where the coefficient of a_r is a polynomial in descending powers of r. The last term in the square brackets will be of order $O(n^3)$,

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* W. E. Byerly, *Fourier Series*, 1893, p. 151. See also N. Nielsen, *Théorie des Fonctions Métasphériques*, 1911.

† Darboux, loc. cit., p. 378.

‡ This is shown by using equation (1), Darboux, loc. cit., p. 377, and making the transformation $x=(1-\xi)/2$, $x=0$ corresponding to $\xi=1$.

$\delta \leqq -1$, which, when multiplied by $a_r X_r$ and summed, will give a convergent series, for

$$a_r X_r O(n^{\delta}) = O(n^{(p-1)/2-p/2+\delta}) = O(n^{\delta'}),$$

where $\delta' = \delta - 1/2 < -1$.

The terms in (4) free from n are

$$R_0 \equiv -c_0 X_1 + \frac{1}{2+p} c_1 X_0.$$

These carry over without change in the process of summation.

The *remainder* terms in (4) will be

$$R_n \equiv \frac{n+p}{2n+p} c_n X_{n+1} - \frac{n+1}{2n+2+p} c_{n+1} X_n. \tag{5}$$

The relation (4) can now be written in the form

$$x \sum_{r=1}^{n} c_r X_r = R_0 + \sum_{r=1}^{n} U_r X_r + R_n, \qquad (c_r = r^i a_r). \tag{6}$$

4. *Application of Summation* (C, k) *to* (6). Apply Cesàro summation of order k to both sides of (6). Representing by $S_{n,i}^{(k)}$ the kth Cesàro mean of $S_{n,i} = \sum_{r=0}^{n} r^i a_r X_r$, we find after transposing the sum $S_{n,i}^{(k)}$ from the right to the left side of equation (6),

$$(x-1)S_{n,i}^{(k)} = R_0 + b_1(p)S_{n,i-2}^{(k)} + \cdots + b_{i-1}(p)S_{n,0}^{(k)}$$
$$+ [S_n^{(k)} \text{ of a convergent series}] + (C, k) \text{ of } R_n.$$

The order of the terms of R_n is $O(n^{i-1/2})$, so, by Theorem 4, Cesàro summability of order $< i - 1/2$ could not be expected.

Then, applying Cesàro summation of order k, so chosen that $(C, k)R_n \to 0$ as $n \to \infty$, and writing $\lim_{n\to\infty} S_{n,k}^{(k)} = S_{\infty,k}^{(k)}$, we have $S_{\infty,i}^{(k)}$ expressed in terms of R_0, $S_{\infty,i-2}^{(k)}$, $S_{\infty,i-3}^{(k)}$, $\cdots$, $S_{\infty,0}^{(k)}$ and a convergent series. Values of $S_{\infty,i}^{(k)}$ must be calculated successively; we begin with $S_{\infty,0}^{(k)}$, which can be obtained from the generating function (2), and take successive integral values of i, beginning with $i = 1$. As stated in the introduction, two applications of Cesàro summation, which are equivalent to summation (C, k), are then used to make (C, k) $R_n \to 0$ as $n \to \infty$.

5. *Summation* $(C, i-1)$ *Applied to* R_n. The convergence factors for summability (C, k) have the form

$$\frac{\Gamma(k+n-r+1)}{n^k\Gamma(n-r+1)},$$

where n represents the total number of terms in the sum under consideration and r the rank of the particular term to which the convergence factor is applied. One groups the remainder R_n with the nth term of the sum in the right-hand member of (6) so that the nth Cesàro convergence factor will be applied to R_n as well as to U_nX_n. The remainder R_n becomes, after application of summation $(C, i-1)$,

$$R_n^{(i-1)} = \Gamma(i)\left[\frac{(n^2+p)n}{2n+p}X_{n+1} - \frac{(n+1)^i}{(2n+2+p)}\frac{(p+2n+1)}{2\cdot n^{i-1}}X_n\right]a_n, \tag{7}$$

$$R_n^{(i-1)} = \frac{1}{2}a_n\Gamma(i)\left[1+O\left(\frac{1}{n}\right)\right](n+p)(X_{n+1}-X_n). \tag{8}$$

To evaluate this expression as $n\to\infty$, the Christoffel-Darboux identity is used.

6. *Use of the Christoffel-Darboux Identity.* This identity, when modified to conform to the notation of this paper,* becomes

$$\frac{\Gamma(n+p)}{\Gamma(p)\Gamma(n+1)}(n+p)\frac{X_{n+1}-X_n}{x-1} = \sum_{r=0}^{n}\frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)}(2r+p)X_r. \tag{9}$$

Differentiate both sides of (3), multiply throughout by $2t$, and add to (3); we get

$$\sum_{r=0}^{\infty}\frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)}(2r+p)X_rt^r = \frac{p(1-t^2)}{(1-2tx+t^2)^{p/2+1}},$$

* Darboux, loc. cit.; one substitutes (44), p. 46, in (14), p. 381. After changing x and z into $(1-x)/2$ and $(1-z)/2$, respectively, let $\alpha=p$ and $\gamma=(p+1)/2$. Then let $z=1$, so that $Z_n=Z_{n+1}=1$, and simplify.

which one can write in the form

$$\sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r t^r = \frac{\Gamma(p+1)}{(1-2tx+t^2)^{p/2}} \frac{1-t^2}{1-2tx+t^2}, \tag{10}$$

which incidentally checks with well known relations for Tchebychef (trigonometric) polynomials, where $p=0$, and Legendre polynomials, where $p=1$.

From an article by Fejér,* we have

$$\frac{1-t^2}{1-2tx+t^2} = 2\left(\frac{1}{2} + \sum_{r=1}^{\infty} t^r \cos rv\right), \qquad (x = \cos v),$$

which is the generating function of the trigonometric polynomials. Chapman proved† that $(1/2+\sum_{r=1}^{\infty} \cos rv)$ is summable (C, k), $k>0$, to the value zero.

To obtain the order of summability of

$$\sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r,$$

(which is formula (3) for $t=1$), we may use the method of proof given by Chapman‡ for Legendre polynomials, based on obtaining an asymptotic expression for $S_n^{(k)}$ for the above series by the method of Darboux. Let $x=\cos\theta$. Since

$$\frac{1}{(1-2z\cos\theta+z^2)^{p/2}} = \sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r(\cos\theta) z^r,$$

one has

$$\frac{1}{(1-z)^{k+1}(1-2z\cos\theta+z^2)^{p/2}} = \sum_{n=0}^{\infty} S_n^{(k)} z^n.$$

* L. Fejér, *Über die Laplacesche Reihe*, Mathematische Annalen, vol. 67 (1909), pp. 76–109; p. 81.

† S. Chapman, *The general principle of summability*, Quarterly Journal of Mathematics, vol. 43 (1912), pp. 1–52; p. 27.

‡ Ibid., p. 45.

The generating function of the sequence $\{S_n^{(k)}\}$ is consequently

$$\frac{1}{(1-z)^{k+1}(1-2z\cos\theta+z^2)^{p/2}} = \frac{1}{(1-z)^{k+1}(1-e^{i\theta}z)^{p/2}(1-e^{-i\theta}z)^{p/2}}.$$

This function may (for $0<\theta<\pi$) be developed into a power series in z with unit radius of convergence. If $k+1>p/2$, the predominant singularity on the circle of convergence is at $z=1$. Therefore, the leading term in the asymptotic expression of $S_n^{(k)}$ is given by the coefficient of z^n in the expansion of

$$\frac{1}{(1-e^{i\theta})^{p/2}(1-e^{-i\theta})^{p/2}(1-z)^{k+1}} \equiv \frac{1}{[2(1-\cos\theta)]^{p/2}} \sum_{n=0}^{\infty} A_n^{(k)} z^n.$$

Hence

$$S_n^{(k)} = \frac{1}{[2(1-x)]^{p/2}} A_n^{(k)}(1+o(1)), \quad (x=\cos\theta),$$

$$\lim_{n\to\infty} \frac{S_n^{(k)}}{A_n^{(k)}} = \frac{1}{[2(1-x)]^{p/2}}, \qquad (-1<x<1),$$

so that the series is summable (C, k) for $k+1>p/2$.

7. *Second Application of Cesàro Summation to R_n.* After application of summation $(C, i-1)$, the remainder terms (5) take the form (8), of which the highest ordered part is, from (9),

$$(11)\quad \begin{aligned}&(1/2)\Gamma(i)a_n(n+p)(X_{n+1}-X_n)\\ &= (1/2)\Gamma(i)a_n \frac{\Gamma(n+1)}{\Gamma(n+p)}(x-1)\sum_{r=0}^{n} \frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r.\end{aligned}$$

From (10), if we let $t=1$, and make use of the equation (3) and the equation below (10), we find

$$\sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r = \text{formal Cauchy product}$$

$$\left[\Gamma(p+1)\sum_{r=0}^{\infty} \frac{\Gamma(r+p)}{\Gamma(p)\Gamma(r+1)} X_r\right] \cdot \left[1+2\sum_{r=1}^{\infty}\cos rv\right],$$

and this is summable (C, f) by Theorem 1 (combined with the above result of Chapman) to the value

$$\frac{1}{[2(1-x)]^{p/2}} \cdot 0 = 0,$$

where $f=(p/2-1)+k+1>p/2$, (for $k>0$). Hence, from (11),

$$(1/2)\Gamma(i)a_n \frac{\Gamma(n+1)}{\Gamma(n+p)}(x-1)\sum_{r=0}^{n}\frac{\Gamma(r+p)}{\Gamma(r+1)}(2r+p)X_r$$
$$= O(n^{(p-1)/2+1-p})o(n^f) = o(n^{f+(1-p)/2}),$$

by Theorem 4, $f>p/2$. Now apply summation (C, j). The convergence factor multiplying $R_n^{(j-1)}$ is

$$\frac{\Gamma(j+1)}{n^j}(1+o(1)),$$

so that the remainder terms (5) now become of order

$$o(n^{f+(1-p)/2})O(n^{-j}) = o(n^{f+(1-p)/2-j}) = o(1),$$

if $f+(1-p)/2-j\leqq 0$, that is, $j>1/2$.

Thus the two applications of Cesàro summability $(C, i-1)$ and $(C, j>1/2)$, which are seen to be equivalent to summability $(C, k>i-1/2)$ by Theorem 5, cause the highest ordered part of the remainder terms (5), R_n, to approach zero as $n\to\infty$. The other parts of (8), being of lesser order, likewise approach zero by application of summability $(C, k>i-1/2)$. The value of $S_{\infty,i}^{(k>i-1/2)}$ can now be calculated as indicated in §4.

8. *Legendre Polynomials.* By letting $p=1$, one can easily obtain the values of $S_{\infty,i}^{(k)}$. In this case $a_n=1$ and $\sum_0^\infty X_n = 1/(2-2x)^{1/2}$. The remainder terms are handled by the use of Christoffel's formula

$$(12) \qquad \sum_0^n (2i+1)X_i = (n+1)\frac{X_{n+1}-X_n}{x-1},$$

the series $\sum_0^\infty (2n+1)X_n$ having been previously proved summable $(C, k>1/2)$ to the value zero by Chapman.* The results obtained by this method check with those obtained by Brenke†

* Chapman, loc. cit., p. 46.

† Brenke, loc. cit.

with the Hölder method of summability, where $i = 1, 2, 3$. In the latter case, for example,

$$(x-1)S_{\infty,3}^{(k)} = 2S_{\infty,1}^{(k)} + \frac{1}{4}S_{\infty,0}^{(k)} + \frac{1}{3} - \frac{1}{4}\left[(C,k) \text{ of } \sum_{1}^{\infty} \frac{1}{(2r-1)(2r+3)} X_r\right], \quad (k > 5/2).$$

THE UNIVERSITY OF NEBRASKA

TRIANGULATION OF THE MANIFOLD OF CLASS ONE*

BY S. S. CAIRNS

1. *Introduction.* In the present note, the writer shows that the triangulation method developed in an earlier paper† can be applied to divide a manifold of class one, as defined by Veblen and Whitehead,‡ into the cells of a complex. The manifold of class one includes the regular r-manifold of class C^n on a Riemannian space.§

2. *The Triangulation Theorem.* Let M_r be an arbitrary r-manifold of class one. A *coordinate system* is a correspondence between a point set, the *domain* of the system, on M_r, and a point set, called the *arithmetic domain*, in affine r-space. *Allowable coordinate systems* are a class of one-to-one correspondences whose properties are specified by axioms.‖

THEOREM. *If an r-manifold, M_r, of class one is covered by the domains of a finite set of allowable coordinate systems, it can be triangulated into the cells of a finite complex. Otherwise it can be triangulated into the cells of an infinite complex.*

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* Presented to the Society, December 28, 1934.

† *On the triangulation of regular loci*, Annals of Mathematics, vol. 35 (1934), pp. 579–587. Hereafter we refer to this paper as Triangulations.

‡ *A set of axioms for differential geometry*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 551–561; also, *The Foundations of Differential Geometry*, Cambridge Tract No. 29, 1932, Chapter 6, referred to below as Foundations.

§ Marston Morse, *The Calculus of Variations in the Large*, Colloquium Publications of this Society, vol. 18 (1934), Chapter 5.

‖ Veblen and Whitehead, loc. cit.

We will discuss only the finite case. The other can then be treated by a ready extension.

3. *The Coordinates H.* LEMMA. *If M_r is covered by the domains of a finite set of allowable coordinate systems, then it can be covered by a finite set $(\gamma)\equiv(\gamma^1, \cdots, \gamma^n)$ of closed r-cells, where γ^i, $(i=1, \cdots, n)$, is on the domain of an allowable coordinate system H^i, and where the arithmetic domain of H^i is an r-simplex, a^i.*

For, let δ denote one of a finite set of domains covering M_r. Let δ' be the part of δ not covered by the other domains of the set. Since the domains are regions (Foundations, p. 78, Theorem 4), there exists, by the Heine-Borel theorem, a finite set of r-cells on δ covering δ'. Hence M_r can be covered by a finite set of r-cells $(\alpha)\equiv(\alpha^1, \cdots, \alpha^n)$, where (1) α^i, $(i=1, \cdots, n)$ is the domain of an allowable coordinate system H^i and (2) the arithmetic domain of H^i is a simplex (Foundations, p. 76, Theorem 2). It remains only to choose γ^i as a closed r-cell on α^i containing all points not covered by the other cells of the set (α).

(A) *We will denote by β^i, $(i=1, \cdots, n)$, an r-cell which contains γ^i and whose closure, $\bar{\beta}^i$, lies on α^i.*

An m-cell, $(m=0, \cdots, r)$, on α^i is called an H_m^i-cell if its image under H^i is a simplex. Now let σ_{m-1} be an $H_{m-1}^{i_1\cdots i_p}$-cell, assuming the term defined. If a point P can be joined to each point of σ_{m-1} by an H_1^j-cell, $(j\neq i_1, \cdots, i_{p-1})$, and if the totality of such cells is an m-cell, σ_m, then σ_m is called an $H_m^{i_1\cdots i_p j}$-cell or an $H_m^{i_1\cdots i_p}$-*cell* according as $j\neq i_p$ or $j=i_p$. A set of $(r+1)$ points will be called $(j_1\cdots j_q)$-*dependent* if, using them in any order whatever, we can construct every conceivable $H_r^{i_1\cdots i_p}$-cell, where $(j_1\cdots j_q)\supset(i_1\cdots i_p)$.

(B) *The intersection $\alpha^j\cdot\alpha^k$ corresponds under H^j and H^k to subsets of a^j and a^k, respectively. The resultant of H^j and H^k is a homeomorphism*

$$H^{jk}: \qquad v_i = f_i(u), \qquad (i=1, \cdots, r), \tag{1}$$

between these subsets. By Axiom A_1, since the H's are allowable, H^{jk} is of class one (each f is continuous with its first derivatives) and regular (the jacobian of the f's is nowhere zero).*

* Veblen and Whitehead, loc. cit.

The α's, β's, and γ's play the role of the cells similarly denoted in Triangulations, §3, and it is now possible to read §§4–7 of that paper, omitting §4, Lemma 1, and §4 (A), but making no other essential changes. We accordingly confine ourselves to a summary of the work, referring to Triangulations for details.

The vertices of an r-simplex constitute a *θ-set* if every angle* between a bounding $(r-1)$-simplex and an edge to the opposite vertex exceeds θ.

(C) *Let (P) be a set of $(r+1)$ points on M_r and let $(\beta^{i_1}, \cdots, \beta^{i_q})$ be the set of all the cells β [see* (A) *above] each containing a point of (P). Then, for any $\theta>0$, a distance $\rho(\theta)>0$ exists, so small that if (P) is a θ-set of diameter less than $\rho(\theta)$ in terms of some H^i-distance and H^i-angles, $(i=j_1, \cdots, j_q)$, then (P) is $(j_1 \cdots j_q)$-independent.* [Triangulations, §5(D) and §6, Theorem.]

4. *Construction of the Triangulation.* Let the r-simplex a^1, the arithmetic domain of H^1, be triangulated into simplexes, such that the vertices of each r-simplex constitute a θ_1-set of diameter less than $\rho(\theta_1)$, for some $\theta_1>0$. The cells which correspond under H^1 to these simplexes constitute a triangulation (σ^1) of a^1 into H^1-cells.

As the basis for a recurrent process we assume that we have, for some number j of the set $(1, \cdots, n-1)$, a triangulation (σ^j) of a neighborhood on M_r of $(\gamma^1+\cdots+\gamma^j)$, where (1) the cells (σ^j) are $H^{i_1 \cdots i_p}$-cells, $(i_1<i_2<\cdots<i_p \leqq j)$, and (2) the vertices of each r-cell of (σ^j) satisfy condition (C) in §3 for some angle of a set $(\theta_1, \cdots, \theta_j)$.

The $(j+1)$th step of the process modifies (σ^j) and extends it to cover a neighborhood of γ^{j+1}. Let P denote an interior vertex† of (σ^j) on $\bar{\beta}^{j+1}$. Let σ_m be any m-cell of the star of P, and let σ_{m-1} be the bounding $(m-1)$-cell of σ_m opposite P. By the hypotheses of the recurrency, σ_{m-1} is an $H_{m-1}^{i_1 \cdots i_p}$-cell for some $(i_1, \cdots, i_p)$ and, applying §3(C), P and σ_{m-1} determine an $H_m^{i_1 \cdots i_p j+1}$-cell σ_m'. Let σ_m' replace σ_m as σ_m denotes successively the various cells of the star of P. The totality of these replacements will be referred to as the *introduction of H^{j+1}-straightness*

* We define *distance* and *angle* by regarding the coordinates in arithmetic space as a rectangular cartesian system.

† A vertex of (σ^j) is called *exterior* if it is on the closure of $M_r-(\sigma^j)$. Otherwise it is called *interior*.

on the star of P. When H^{i+1}-straightness is thus introduced on the star of an interior vertex, a new complex is obtained covering the same points as (σ^i) [Triangulations, §7(C)].

Now let H^{i+1}-straightness be introduced successively on the stars of the other interior vertices of (σ^i) on $\bar{\beta}^{i+1}$. When this is done, let all cells with exterior vertices be dropped.* In the resulting complex,† $(\sigma^{i+1})^0$, every cell with all its vertices on $\bar{\beta}^{i+1}$ is an H^{i+1}-cell. Now consider a complex $(\sigma^{i+1})''$ made up of H^{i+1}-cells on β^{i+1}, such that (1) $(\sigma^{i+1})''$ covers the part of γ^{i+1} not on $(\sigma^{i+1})^0$ and (2) all overlapping cells of $(\sigma^{i+1})^0$ and $(\sigma^{i+1})''$ are H^{i+1}-cells.‡ By elementary subdivisions, let $(\sigma^{i+1})^0$ and $(\sigma^{i+1})''$ be subdivided until they have just a subcomplex in common. The set (σ^{i+1}) of all cells each in one of these subdivided complexes now satisfies all the requirements of the recurrent process, save perhaps the one involving §3(C). We can satisfy this one also by slightly displacing certain vertices in the elementary subdivisions above and imposing a new upper bound on the cells employed. This completes an outline of the $(j+1)$th step.

The nth step completes the triangulation of M_r into the cells of a simplicial complex.

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* Because the domains $(\alpha^i, \beta^i, \gamma^i)$ are nested [§3(A)], it is possible, by imposing an upper bound on the diameters of the cells employed, to ensure that the dropping of cells with exterior vertices will not uncover any points of a certain neighborhood of $(\gamma^1+\cdots+\gamma^i)$. For details, see Triangulations, §7(A) and (B).

† See Triangulations, §7, Lemma 2, for a proof that $(\sigma^{i+1})^0$ is a complex. The present construction of $(\sigma^{i+1})^0$ is suggested on page 587 at the end of the proof.

‡ For further restrictions on $(\sigma^{i+1})''$ see Triangulations, under §7(D).

DIFFERENTIATION OF SEQUENCES*

BY ORRIN FRINK, JR.

Conditions under which one may differentiate term by term a convergent sequence of functions are to be found in the literature.† Some of these conditions are obtained as corollaries of theorems about termwise integration, and consequently contain in the hypothesis the assumption that the derived sequence converges. Without this assumption, sufficient conditions for termwise differentiability may be obtained from the fundamental theorem on reversing the order of iterated limits. This possibility is mentioned by Hobson (vol. 2, p. 336), but he does not obtain the theorems of the present paper.

By this means it may be shown that the equicontinuity of the functions of the derived sequence is sufficient for termwise differentiability, but the condition of equicontinuity is stronger than necessary for mere convergence, since it implies also that the derivatives are continuous and converge uniformly to a continuous function. This paper is concerned with weaker conditions, for which the names *normal*, *uniform*, *almost normal*, *almost uniform* are used. These conditions, like that of equicontinuity, are related also to the question of the compactness of sets of functions for different types of convergence. The procedure that will be followed is to state the principal lemma, define the different conditions, and then show their relations, first to compactness, and then to termwise differentiation.

LEMMA. *If the sequence of functions* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on the closed interval* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist at the point* x_0 *of* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x_0)$ *exist at* x_0 *and be the limit of the sequence of derivatives* (ii) $\{f_n'(x_0)\}$ *is that for every* $\epsilon>0$ *there exists a* $\delta>0$ *such that if* $|x-x_0|<\delta$, *then*

$$\left|\frac{f_n(x)-f_n(x_0)}{x-x_0}-f_n'(x_0)\right|<\epsilon \tag{1}$$

* Presented to the Society, December 27, 1934.

† Hobson, *The Theory of Functions of a Real Variable*, 2nd ed., vol. 2, pp. 332–338.

for almost all n (that is, for all except a finite number of values of n).

This is a special case of a general theorem on reversing the order of iterated limits,* which may be stated as follows:

If $\lim_{x\to x_0} F_n(x) = F_n$, *and* $\lim_{n\to\infty} F_n(x) = F(x)$ *for* $x \neq x_0$, *then* $\lim_{x\to x_0}\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} F_n(x)$ *if and only if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* $|x - x_0| < \delta$, *then* $|F_n(x) - F_n| < \epsilon$ *for almost all* n.

The lemma follows from this if we let

$$F_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0} \quad \text{and} \quad F_n = f_n'(x_0).$$

DEFINITIONS. The sequence (i) $\{f_n(x)\}$ is said to be *normal*† at the point x_0 of $[a, b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ and an N such that if $|x - x_0| < \delta$ and $n > N$, then $|f_n(x) - f_n(x_0)| < \epsilon$.

The sequence (i) is said to be *almost normal* at x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f_n(x) - f_n(x_0)| < \epsilon$ for almost all n.

The sequence (i) is said to be *uniform* at x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ and an N such that if $|x - x_0| < \delta$ and $m > N$ and $n > N$, then $|f_m(x) - f_m(x_0) - f_n(x) + f_n(x_0)| < \epsilon$.

The sequence (i) is said to be *almost uniform* at x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f_m(x) - f_m(x_0) - f_n(x) + f_n(x_0)| < \epsilon$ for all m, n sufficiently large.

Note that in the definitions of *normal* and *uniform* the value of N is selected in advance of the value of x. Note also these relationships: if a sequence is equicontinuous, it is normal, and if it is normal, it is both uniform and almost normal, and if it is either uniform or almost normal, it is almost uniform. Note also that (i) is uniform if and only if the double sequence (ii) $\{f_m(x) - f_n(x)\}$ is normal, and (i) is almost uniform if and only if (ii) is almost normal, when the definitions are extended to double sequences.

The following results concerning normal sequences will be restated from Hahn and Carathéodory:

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* Hobson, vol. 1, p. 385; Hahn, *Theorie der Reellen Funktionen*, 1st ed., pp. 288, 294.

† Hahn, 2nd ed., p. 229; Carathéodory, Mathematische Annalen, vol. 101 (1929), p. 515.

A necessary and sufficient condition that every subsequence of (i) $\{f_n(x)\}$ *contain in turn a subsequence which converges continuously (that is, uniformly to a continuous function) on* $[a, b]$ *is that* (i) *be normal at every point of* $[a, b]$ *and bounded in n for every x. If* (i) *converges at* x_0, *it converges continuously at* x_0 *if and only if it is normal at* x_0.

We shall prove analogous results for the other conditions.

THEOREM 1. *If the sequence* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *it is almost uniform at every point of* $[a, b]$. *Furthermore, it converges uniformly on* $[a, b]$ *if and only if it is uniform at every point of* $[a, b]$; *and the limit function* $f(x)$ *is continuous at a point* x_0 *of* $[a, b]$ *if and only if the sequence* (i) *is almost normal at* x_0.

Since (i) converges on $[a, b]$, given $\epsilon>0$ and two points x and x_0 of $[a, b]$, there exists an N such that for all $m>N$, $n>N$

$$|f_m(x) - f_m(x_0) - f_n(x) + f_n(x_0)| < \epsilon. \tag{2}$$

Hence (i) is almost uniform at every point x_0 of $[a, b]$.

Now if (i) is uniform at x_0, given $\epsilon>0$ select $\delta>0$ and N' such that if $|x-x_0|<\delta$ and $m>N'$, $n>N'$, then (2) holds. Since (i) converges at x_0, select $N>N'$ such that if $m>N$, $n>N$, then

$$|f_m(x_0) - f_n(x_0)| < \epsilon. \tag{3}$$

Adding (2) and (3), we have for $|x-x_0|<\delta$, $m>N$, $n>N$,

$$|f_m(x) - f_n(x)| < 2\epsilon. \tag{4}$$

Hence the convergence is locally uniform* at every point x_0 of $[a, b]$. Since $[a, b]$ is a closed interval, the convergence is uniform on $[a, b]$.

Conversely, if (1) converges uniformly on $[a, b]$ then for every $\epsilon>0$ an N exists such that for any two points x and x_0 of $[a, b]$, we have, for $m>N$, $n>N$,

$$|f_m(x) - f_m(x_0) - f_n(x) + f_n(x_0)| < \epsilon. \tag{5}$$

Hence the sequence is uniform at every point x_0 of $[a, b]$.

Next, if (1) is almost normal at x_0, given $\epsilon>0$ select $\delta>0$ such that if $|x-x_0|<\delta$, then for almost all n

$$|f_n(x) - f_n(x_0)| < \epsilon. \tag{6}$$

* Hahn, 2nd ed., pp. 214, 216.

Since (1) converges at x and x_0, we have for almost all n

$$|f_n(x_0) - f(x_0)| < \epsilon, \qquad |f_n(x) - f(x)| < \epsilon. \tag{7}$$

Adding (6) and (7), we have, if $|x-x_0| < \delta$,

$$|f(x) - f(x_0)| < 3\epsilon. \tag{8}$$

Hence $f(x)$ is continuous at x_0. Conversely, if $f(x)$ is continuous at x_0, given $\epsilon > 0$ select $\delta > 0$ such that if $|x-x_0| < \delta$, we have

$$|f(x_0) - f(x)| < \epsilon. \tag{9}$$

Since (1) converges at x and x_0, then (7) holds for almost all n. Adding (7) and (9) we have, if $|x-x_0| < \delta$, for almost all n

$$|f_n(x_0) - f_n(x)| < 3\epsilon. \tag{10}$$

Hence (1) is almost normal at x_0.

THEOREM 2. *If the sequence* (i) $\{f_n(x)\}$ *is almost uniform at every point of* $[a, b]$ *and bounded in n for every x, then every subsequence of* (i) *contains in turn a subsequence* (ii) $\{f_{n_k}(x)\}$ *which converges on* $[a, b]$ *to some function* $f(x)$. *Furthermore, if* (i) *is uniform at every point of* $[a, b]$, *the convergence of* (ii) *is uniform on* $[a, b]$, *and if* (i) *is almost normal at every point of* $[a, b]$, *the limit function* $f(x)$ *of* (ii) *is continuous on* $[a, b]$.

Since (i) is bounded in n for every x, from every subsequence of (i) a subsequence (ii) $\{f_{n_k}(x)\}$ may be chosen* which converges at every rational point of $[a, b]$. Now let x_0 be any irrational point of $[a, b]$. Since (i) is almost uniform at x_0, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x-x_0| < \delta$, then

$$|f_{n_k}(x) - f_{n_k}(x_0) - f_{n_l}(x) + f_{n_l}(x_0)| < \epsilon \tag{11}$$

for all k, l sufficiently large. Choose x rational. Then (ii) converges at x, and for k and l sufficiently large we have

$$|f_{n_k}(x) - f_{n_l}(x)| < \epsilon. \tag{12}$$

From (11) and (12), we get

$$|f_{n_k}(x_0) - f_{n_l}(x_0)| < 2\epsilon. \tag{13}$$

Hence (ii) converges also at x_0, and therefore at every point of $[a, b]$. The second part of the theorem follows from Theorem 1.

* Hahn, 2nd ed., p. 230.

THEOREM 3. *If the sequence* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on the closed interval* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist on* $[a, b]$ *and the sequence of derivatives* (ii) $\{f_n'(x)\}$ *is uniform at a point* x_0 *of* $[a, b]$, *then the derivative* $f'(x_0)$ *exists at* x_0, *and the derived sequence* (ii) *converges to* $f'(x_0)$ *at* x_0.

Given $\epsilon>0$, select $\delta>0$ and N in accordance with the condition that (ii) is uniform at x_0, and let $|x-x_0|<\delta$. Then by the mean value theorem, applied to the function $f_m(x)-f_n(x)$, and the condition that (ii) is uniform at x_0, we have, for $m>N$, $n>N$,

$$(14)\quad \left| \frac{f_m(x)-f_m(x_0)}{x-x_0} - f_m'(x_0) - \frac{f_n(x)-f_n(x_0)}{x-x_0} + f_n'(x_0) \right| = |f_m'(x') - f_m'(x_0) - f_n'(x') + f_n'(x_0)| < \epsilon,$$

where x' may vary with m and n, but always lies between x_0 and x.

Now by the definition of derivative, for any fixed $n>N$ there exists a positive $\delta'<\delta$ such that for $|x-x_0|<\delta'$ we have

$$(15)\quad \left| \frac{f_n(x)-f_n(x_0)}{x-x_0} - f_n'(x_0) \right| < \epsilon.$$

Adding (14) and (15), we have, if $|x-x_0|<\delta'$ and $m>N$,

$$(16)\quad \left| \frac{f_m(x)-f_m(x_0)}{x-x_0} - f_m'(x_0) \right| < 2\epsilon.$$

Hence the condition of the lemma is fulfilled, and the conclusion of the theorem follows.

COROLLARY A. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x)$ *exist on* $[a, b]$ *and that the derived sequence* (ii) $\{f_n'(x)\}$ *converge uniformly to* $f'(x)$ *on* $[a, b]$, *is that* (ii) *be uniform at every point of* $[a, b]$.

COROLLARY B. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *then a necessary and sufficient condition that* $f'(x)$ *exist and be continuous on* $[a, b]$, *and that the derived sequence* (ii) $\{f_n'(x)\}$ *converge uniformly to* $f'(x)$ *on* $[a, b]$, *is that* (ii) *be normal at every point of* $[a, b]$.

COROLLARY C. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *and the functions of the derived sequence* (ii) $\{f_n'(x)\}$ *are continuous on* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x)$ *exist, and that the sequence* (ii) *converge uniformly to* $f'(x)$ *on* $[a, b]$, *is that the derivatives* $f_n'(x)$ *be equicontinuous at every point of* $[a, b]$.

COROLLARY D. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$ *and the second derivatives* $f_n''(x)$ *exist and are uniformly bounded on* $[a, b]$, *then* $f'(x)$ *exists and the derived sequence* (ii) $\{f_n'(x)\}$ *converges uniformly to* $f'(x)$ *on* $[a, b]$.

Corollary D follows from Corollary C, since the first derivatives are equicontinuous if the second derivatives are uniformly bounded. The others follow from Theorems 1 and 3, the relations mentioned above between the different conditions, and the properties of equicontinuous and normal sequences.

THEOREM 4. *If the sequence* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist and are summable on* $[a, b]$, *and the derived sequence* (ii) $\{f_n'(x)\}$ *converges in the mean on* $[a, b]$, *that is,* $\int_a^b |f_m'(x) - f_n'(x)|\,dx \to 0$, *then a necessary and sufficient condition that the derivative* $f'(x)$ *exist and be continuous on* $[a, b]$ *and that the sequence* (ii) *converge to* $f'(x)$ *on* $[a, b]$ *is that* (ii) *be almost normal at every point of* $[a, b]$.

If (ii) is almost normal at x_0, given $\epsilon > 0$ select $\delta > 0$ in accordance with the definition of almost normal, and let $|x - x_0| < \delta$. Let E_k denote the set of values of t between x_0 and x such that for some $n > k$,

$$|f_n'(t) - f_n'(x_0)| \geqq \epsilon, \tag{17}$$

and let C_k be the complement of E_k with respect to the interval $[x_0, x]$. Since $|t - x_0| < \delta$, and (ii) is almost normal at x_0, no value of t is a member of more than a finite number of the sets E_k. Hence the measure of E_k approaches zero. Now

$$\left|\int_{x_0}^{x} [f_n'(t) - f_n'(x_0)]\,dt\right| \leqq \int_{C_k} |f_n'(t) - f_n'(x_0)|\,dt + \int_{E_k} |f_n'(t)|\,dt + \int_{E_k} |f_n'(x_0)|\,dt. \tag{18}$$

If $n>k$, the first integral on the right of (18) is less than $\epsilon|x-x_0|$, since on C_k the integrand is $<\epsilon$. By the convergence in the mean, we may select m such that, for $n>m$,

$$\int_a^b |f_m'(t) - f_n'(t)|\,dt < \frac{\epsilon}{2}|x - x_0|. \tag{19}$$

Since the measure of E_k approaches zero, $k>m$ can be selected such that

$$\int_{E_k} |f_m'(t)|\,dt < \frac{\epsilon}{2}|x - x_0|, \tag{20}$$

since m is now fixed, and the integral in (20) is absolutely continuous. Hence from (19) and (20), for $n>k$,

$$\int_{E_k} |f_n'(t)|\,dt < \epsilon|x - x_0|. \tag{21}$$

But this is the second integral on the right of (18). Now the sequence $\{f_n'(x_0)\}$ is bounded, for if it were not, there would exist a subsequence of (ii) which diverged to infinity at every point of some neighborhood of x_0, since (ii) is almost normal at x_0. But this is impossible, since (ii) converges in the mean, and every subsequence contains in turn a subsequence which converges almost everywhere. Because of the boundedness, a k exists such that for $n>k$ the third integral on the right of (18) is less than $\epsilon|x-x_0|$. Combining these results, we see that a k exists such that for $n>k$,

$$\left|\int_{x_0}^{x} [f_n'(t) - f_n'(x_0)]dt\right| < 3\epsilon|x - x_0|. \tag{22}$$

Since $f_n'(x)$ is summable, there results on dividing (22) by $|x-x_0|$, for $n>k$,

$$\left|\frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0)\right| < 3\epsilon. \tag{23}$$

This is the condition of the lemma; therefore $f_n'(x)$ exists, and (ii) converges to it at x_0, and hence at every point of $[a, b]$. By Theorem 1, $f'(x)$ is continuous on $[a, b]$. Conversely, if $f'(x)$ is continuous on $[a, b]$ and (ii) converges to it, by Theorem 1 the sequence (ii) is almost normal at every point of $[a, b]$.

THEOREM 5. *If* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist and are summable on* $[a, b]$, *and the derived sequence* (ii) $\{f_n'(x)\}$ *converges in the mean on* $[a, b]$, *then a necessary and sufficient condition that the sequence* (ii) *converge on* $[a, b]$ *to some function* $g(x)$ (*not necessarily equal to* $f'(x)$) *is that* (ii) *be almost uniform at every point of* $[a, b]$.

If (ii) is almost uniform at x_0, given $\epsilon>0$, let us select $\delta>0$ in accordance with the definition of almost uniform, and let $|x-x_0|<\delta$. By the same method of proof that was used in Theorem 4, we may show that a k exists, such that if $m>k$, $n>k$,

$$(24)\qquad \left|\int_{x_0}^{x}[f_m'(t)-f_m'(x_0)-f_n'(t)+f_n'(x_0)]dt\right|<3\epsilon\,|x-x_0|.$$

In fact, since (ii) is almost uniform, the double sequence $\{f_m'(x)-f_n'(x)\}$ is almost normal, and (24) corresponds to formula (22) in the proof of Theorem 4, applied to this double sequence. Dividing (24) by $|x-x_0|$, we have, for $m>k$, $n>k$,

$$(25)\qquad \left|\frac{f_m(x)-f_m(x_0)}{x-x_0}-f_m'(x_0)-\frac{f_n(x)-f_n(x_0)}{x-x_0}+f_n'(x_0)\right|<3\epsilon.$$

Since (i) converges at x and x_0, an $N>k$ exists such that if $m>N$, $n>N$,

$$(26)\qquad \left|\frac{f_n(x)-f_n(x_0)}{x-x_0}-\frac{f_m(x)-f_m(x_0)}{x-x_0}\right|<\epsilon.$$

Adding (25) and (26), we have, for $m>N$, $n>N$,

$$(27)\qquad |f_m'(x_0)-f_n'(x_0)|<4\epsilon.$$

Therefore (ii) converges at x_0, and hence at every point of $[a, b]$. Conversely, if (ii) converges on $[a, b]$, by Theorem 1 it is almost uniform at every point of $[a, b]$. That the limit function of (ii) need not be $f'(x)$ can be seen from an example. Let (i) be the sequence $\{x^n/n\}$ on the interval $[0, 1]$. Then the derived sequence converges on $[0, 1]$, but not to the derivative of the limit function when $x=1$.

THE PENNSYLVANIA STATE COLLEGE

ON THE SINGULARITIES OF AN ANALYTIC FUNCTION*

BY E. W. MILLER

1. *Introduction.* We shall consider an analytic function $f(z)$ represented by the series $\sum_{n=0}^{\infty} a_n z^n$ whose circle of convergence we shall suppose for simplicity to be the *unit* circle with center at the origin of the complex plane. Our purpose is to give simple generalizations of certain theorems of Pringsheim and Hadamard relative to the singularities of $f(z)$ on C, the circumference of the circle of convergence.

With the present hypotheses and notation the theorems in question may be formulated as follows:

THEOREM OF PRINGSHEIM.† *In order that $z=1$ be a simple pole of $f(z)$ and that there be no further singularity of $f(z)$ on C it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \left| a_{n+1} - a_n \right|^{1/n} < 1.$$

THEOREM OF HADAMARD.‡ *In order that there be just one simple pole and no further singularity of $f(z)$ on C it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \left| a_n^2 - a_{n-1}a_{n+1} \right|^{1/n} < 1.$$

2. *Generalizations of the Above Theorems.* We shall first establish a generalization of Pringsheim's theorem.

THEOREM 1. *In order that $z=1$ be a pole of order m of $f(z)$ and that there be no further singularity of $f(z)$ on C it is necessary and sufficient that there exist a polynomial $g(x)$ of degree $m-1$ such that, if we put $A_n = a_n/g(n)$, we have*

$$\overline{\lim_{n\to\infty}} \left| A_{n+1} - A_n \right|^{1/n} < 1.$$

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* Presented to the Society, April 20, 1935.

† A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre*, vol. 2, part 2, 2d ed., 1932, p. 916.

‡ See, for instance, P. Dienes, *The Taylor Series*, p. 333.

PROOF. Suppose that $z=1$ is a pole of order m of $f(z)$ and that there is no further singularity of $f(z)$ on C. Then $f(z)=f_1(z)+f_2(z)$, where $f_1(z)$ has $z=1$ as a pole of order m and has no other singularity in the complex plane, and where $f_2(z)$ has no singularity within or on C. According to a theorem of Leau and Wigert* there is a polynomial $g(x)$ of degree $m-1$ such that $f_1(z)=\sum_{n=0}^{\infty}g(n)z^n$. We choose this as the polynomial $g(x)$ of Theorem 1. We have then $a_n=g(n)+b_n$, where $\overline{\lim}_{n\to\infty}|b_n|^{1/n}<1$. Hence $A_n=1+B_n$, where $B_n=b_n/g(n)$. Clearly $\overline{\lim}_{n\to\infty}|B_n|^{1/n}<1$. It follows at once that $\overline{\lim}_{n\to\infty}|A_{n+1}-A_n|^{1/n}<1$.

To prove the condition sufficient we notice that by the theorem of Pringsheim cited above $\sum_{n=0}^{\infty}A_nz^n$ has $z=1$ as a simple pole and has no further singularity on C. Thus

$$\sum_{n=0}^{\infty}A_nz^n = k\sum_{n=0}^{\infty}z^n + \sum_{n=0}^{\infty}c_nz^n, \quad \text{where } \overline{\lim_{n\to\infty}}\,|c_n|^{1/n} < 1.$$

Therefore

$$\sum_{n=0}^{\infty}a_nz^n = \sum_{n=0}^{\infty}g(n)A_nz^n = k\sum_{n=0}^{\infty}g(n)z^n + \sum_{n=0}^{\infty}g(n)c_nz^n.$$

Now by the Leau-Wigert theorem $\sum_{n=0}^{\infty}g(n)z^n$ has $z=1$ as a pole of order m and has no other singularity in the complex plane. Furthermore $\overline{\lim}_{n\to\infty}|g(n)c_n|^{1/n}<1$. Hence $\sum_{n=0}^{\infty}a_nz^n$ has $z=1$ as a pole of order m and has no further singularity on C.

In case the type of pole is specified it may be easy to determine the polynomial $g(x)$. For example, suppose we have $f(z)=(k/(1-z)^m)+F(z)$, where the Maclaurin series for $F(z)$ has a radius of convergence >1. Since

$$1/(1-z)^m=\frac{1}{(m-1)!}\sum_{n=0}^{\infty}(n+m-1)\cdots(n+1)z^n,$$

we may put $g(n)=(k/(m-1)!)\ (n+m-1)\cdots(n+1)$. After simplification our condition that $f(z)$ be of the above described type assumes the form

$$\overline{\lim_{n\to\infty}}\,|(n+m)a_n-(n+1)a_{n+1}|^{1/n}<1.$$

That this condition is not merely necessary but also sufficient follows easily once we put $a_n=(n+m-1)\cdots(n+1)\,k_n$.

* See, for instance, Dienes, loc. cit., pp. 337–339.

The theorem of Hadamard cited in (1) is susceptible of a similar generalization.

THEOREM 2. *In order that there be just one pole of order m and no further singularity of $f(z)$ on C it is necessary and sufficient that there exist a polynomial $g(x)$ of degree $m-1$ such that if we put $A_n = a_n/g(n)$, then*

$$\overline{\lim_{n\to\infty}} \left| A_n^2 - A_{n-1}A_{n+1} \right|^{1/n} < 1.$$

PROOF. Suppose there is a pole of order m and no further singularity of $f(z)$ on C. Let this be the point $z=b$. If we put $z=bw$, the series $\sum_{n=0}^{\infty} a_n b^n w^n$ will have $w=1$ as a pole of order m and will have no other singularity on the circumference of the circle of convergence. Accordingly there exists a polynomial $g(x)$ of degree $m-1$ such that $a_n b^n = g(n) + c_n'$, where $\overline{\lim}_{n\to\infty} |c_n'|^{1/n} < 1$. Hence $A_n = (1/b^n) + c_n$, where $\overline{\lim}_{n\to\infty} |c_n|^{1/n} < 1$. The condition of our theorem now follows easily.

That the condition is sufficient follows by an argument similar to that used in the proof of the sufficiency of the condition in Theorem 1.

Again, it is easy to show that $f(z) = (k/(\alpha - z)^m) + F(z)$, where $|\alpha| = 1$ and where the Maclaurin series for $F(z)$ has a radius of convergence >1, if and only if

$$\overline{\lim_{n\to\infty}} \left| n(n+m)a_n^2 - (n+1)(n+m-1)a_{n-1}a_{n+1} \right|^{1/n} < 1.$$

3. *A Theorem on Limits.* The most important step in the proof of Pringsheim's theorem is to show that, under the conditions of the theorem, $\lim_{n\to\infty} a_{n+1}/a_n = 1$. In our generalization of this theorem it may be noticed that we have

$$\overline{\lim_{n\to\infty}} \left| a_{n+1} - \frac{g(n+1)}{g(n)} a_n \right|^{1/n} < 1$$

and that again $\lim_{n\to\infty} a_{n+1}/a_n = 1$. These facts suggest the following theorem on limits.

THEOREM 3. *If $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 1$, in order that $\lim_{n\to\infty} a_{n+1}/a_n = 1$, it is necessary and sufficient that there exist a sequence of complex numbers λ_n such that*

(1) $\lim_{n\to\infty} \lambda_n = 1$; *and* (2) $\overline{\lim_{n\to\infty}} \left| a_{n+1} - \lambda_n a_n \right|^{1/n} < 1.$

PROOF. The necessity is obvious. We need merely put $\lambda_n = a_{n+1}/a_n$. To prove the sufficiency we notice first that if we can prove that $\lim_{n\to\infty} |a_n|^{1/n}$ exists, then $\lim_{n\to\infty} a_{n+1}/a_n = 1$. For from (2) it would follow that

$$\overline{\lim_{n\to\infty}} \left| a_n \right|^{1/n} \left| \frac{a_{n+1}}{a_n} - \lambda_n \right|^{1/n} < 1,$$

and therefore $\lim_{n\to\infty} a_{n+1}/a_n = 1$. We shall accordingly prove that $\lim_{n\to\infty} |a_n|^{1/n} = 1$. There exists a positive number $k < 1$ such that for n sufficiently large $|a_{n+1} - \lambda_n a_n| < k^n$, so that

$$|a_{n+1} - a_n| < k^n + |\epsilon_n||a_n|,$$

where $\lim_{n\to\infty} \epsilon_n = 0$,

$$|a_{n+2} - a_{n+1}| < k^{n+1} + |\epsilon_{n+1}||a_{n+1}|,$$

and in general

$$|a_{n+j} - a_{n+j-1}| < k^{n+j-1} + |\epsilon_{n+j-1}||a_{n+j-1}|.$$

Therefore

$$|a_{n+j} - a_n| < \frac{k^n}{1-k} + |\epsilon_n||a_n| + \sum_{r=n+1}^{n+j-1} |\epsilon_r||a_r|.$$

Now let us suppose that $\overline{\lim}_{n\to\infty} |a_n|^{1/n} \neq \underline{\lim}_{n\to\infty} |a_n|^{1/n}$. We can find α and β such that $k < \beta < 1$ and $\alpha < \beta$, and two increasing sequences of positive integers $\{p\}$ and $\{q\}$ such that $|a_p| > \beta^p$, $|a_q| < \alpha^q$ and $|a_n| \leqq \beta^n$ if n is not a term of the sequence $\{p\}$. It is obvious that from $\{p\}$ and $\{q\}$ we can select subsequences $\{p_i\}$ and $\{q_i\}$ of such a nature that $p_i < q_i$ and that if $p_i < n < q_i$, then $|a_n| \leqq \beta^n$. We have of course $|a_{p_i}| > \beta^{p_i}$ and $|a_{q_i}| < \alpha^{q_i}$. Accordingly we have $|a_{p_i} - a_{q_i}| > |a_{p_i}| - \alpha^{q_i}$. On the other hand, from our preliminary result we have

$$|a_{p_i} - a_{q_i}| = |a_{q_i} - a_{p_i}| < \frac{k^{p_i}}{1-k} + |\epsilon_{p_i}||a_{p_i}| + \sum_{r=p_i+1}^{q_i-1} |\epsilon_r||a_r|.$$

Therefore

$$|a_{p_i} - a_{q_i}| < \frac{k^{p_i}}{1-k} + |\epsilon_{p_i}||a_{p_i}| + \epsilon'_{p_i} \frac{\beta^{p_i+1}}{1-\beta},$$

where ϵ'_{p_i} denotes the largest of the numbers $|\epsilon_r|$, $(r = p_i+1, \cdots, q_i-1)$. We now consider the difference

$$
\begin{aligned}
&[\,|a_{p_i}| - \alpha^{q_i}] - \left[\frac{k^{p_i}}{1-k} + |\epsilon_{p_i}||a_{p_i}| + \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta}\right] \\
&\quad = |a_{p_i}|(1-|\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta} \\
&\quad > \beta^{p_i}(1-|\epsilon_{p_i}|) - \alpha^{q_i} - \frac{k^{p_i}}{1-k} - \epsilon'_{p_i}\frac{\beta^{p_i+1}}{1-\beta} \\
&\quad = \beta^{p_i}\left(1-|\epsilon_{p_i}| - \frac{\alpha^{q_i}}{\beta^{p_i}} - \frac{k^{p_i}}{\beta^{p_i}(1-k)} - \epsilon'_{p_i}\frac{\beta}{1-\beta}\right).
\end{aligned}
$$

Now for i sufficiently large all the terms within the last parentheses except the first are as small as we please. Hence for sufficiently large i the difference in question is positive. From this contradiction the theorem follows.

In conclusion, we may note as a simple corollary of the above theorem that if $\overline{\lim}_{n\to\infty}|a_n|^{1/n}=1$, then $\lim_{n\to\infty}|a_{n+1}/a_n|=1$ if and only if there exists a sequence of *real* numbers λ_n such that $\lim_{n\to\infty}\lambda_n=1$ and $\overline{\lim}_{n\to\infty}||a_{n+1}|-\lambda_n|a_n||^{1/n}<1$.

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ON THE COEFFICIENTS OF A TYPICALLY-REAL FUNCTION*

BY M. S. ROBERTSON†

1. *Introduction.* It is well known‡ that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

is regular for $|z| \leqq 1$, and if E is defined by the formula

$$E \equiv \underset{|z_1|=|z_2|=1}{\text{maximum}} |\Re f(z_1) - \Re f(z_2)|, \tag{2}$$

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* Presented to the Society, February 23, 1935.

† National Research Fellow.

‡ See E. Landau, Archiv der Mathematik und Physik, (3), vol. 11 (1906), pp. 31–36.

or, in other words, if E is the oscillation of the real part of $f(z)$ for all points z_1 and z_2 on the unit circle, then

$$|a_1| = |f'(0)| \leqq \frac{2}{\pi} E. \tag{3}$$

In this paper an analogous result and extensions are obtained for *all* the coefficients of any function $f(z)$ regular and *typically-real* for $|z|<1$.

DEFINITION. A function $f(z)$, $f(0)=0$, $f'(0)\neq 0$, regular for $|z|<R$, is said to be typically-real with respect to the circle $|z|=R$, if within this circle $f(z)$ is real for, and only for, the points on the real axis.*

It may be noticed, as W. Rogosinski has pointed out, that the class of functions regular and univalent in the circle $|z|=R$ and real on the real axis form a subclass of the class of functions typically-real with respect to this circle.

2. *A Stieltjes Integral Representation for Typically-Real Functions.* Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \text{ real}), \tag{4}$$

be regular and typically-real for $|z|<1$. Then it is known† that $f(z)$ can be represented in the form

$$f(z) = \frac{zg(z)}{1-z^2}, \tag{5}$$

where $g(z)$ is regular for $|z|<1$, $g(0)=1$, $\Re g(z)>0$ for $|z|<1$. Further, by the formula of G. Herglotz,‡ we may write

$$\begin{aligned} g(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\, d\alpha(\theta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-z^2-2iz\sin\theta}{1-2z\cos\theta+z^2}\, d\alpha(\theta), \end{aligned} \tag{6}$$

* See W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Mathematische Zeitschrift, vol. 35 (1932), pp. 93–121.

† See W. Rogosinski, loc. cit., p. 99.

‡ See G. Herglotz, Leipziger Berichte, 1911, pp. 501–511.

where $\alpha(\theta)$ is an odd non-decreasing function of θ in the interval $(-\pi, \pi)$, and where $f(z)$ is real on the real axis, as is also $g(z)$. Hence

$$(7)\qquad \begin{aligned} f(z) &= \frac{1}{\pi}\int_0^\pi \frac{z\,d\alpha(\theta)}{1-2z\cos\theta+z^2} \\ &= \frac{1}{\pi}\int_0^\pi \left(\sum_{n=1}^{\infty}\frac{\sin n\theta}{\sin\theta}z^n\right) d\alpha(\theta). \end{aligned}$$

L. Fejér has observed* that

$$(8)\qquad F(z) = \int_0^z \frac{f(z)}{z}\,dz = z + \sum_2^\infty \frac{a_n}{n}z^n,$$

is univalent and convex in the direction of the imaginary axis for $|z|<1$, that is, no straight line parallel to the imaginary axis can cut the image of the circle $|z|=r$ (for every r in the interval $0<r<1$) mapped by the function $F(z)$ in more than two points. It follows from (7) and (8) by integration that

$$(9)\qquad F(z) = \frac{1}{2\pi i}\int_0^\pi \log\left\{\frac{1-ze^{-i\theta}}{1-ze^{i\theta}}\right\}\frac{d\alpha(\theta)}{\sin\theta}.$$

3. *The Coefficients of a Typically-Real Function.* From (9), since $F(r)$ is real, we have

$$(10)\qquad F(r) = \frac{1}{\pi}\int_0^\pi \arg(1-re^{-i\theta})\frac{d\alpha(\theta)}{\sin\theta}.$$

Since the integrand is an increasing function of r for every θ, we have

$$(11)\qquad \begin{aligned} F(1) \equiv \lim_{r\to 1} F(r) &= \frac{1}{\pi}\int_0^\pi \lim_{r\to 1}\arg(1-re^{-i\theta})\frac{d\alpha(\theta)}{\sin\theta} \\ &= \frac{1}{2\pi}\int_0^\pi \frac{\pi-\theta}{\sin\theta}\,d\alpha(\theta). \end{aligned}$$

Similarly we also have

$$(12)\qquad F(-1) \equiv \lim_{r\to 1} F(-r) = \frac{-1}{2\pi}\int_0^\pi \frac{\theta}{\sin\theta}\,d\alpha(\theta).$$

* See L. Fejér, Journal of the London Mathematical Society, vol. 8 (1933), p. 61, footnote.

These limits are finite or infinite according as the integrals exist or do not exist. Hence if we define

$$E(r) \equiv F(r) - F(-r) = \int_{-r}^{r} \frac{f(t)}{t}\,dt = \int_{-1}^{+1} \frac{f(rt)}{t}\,dt, \tag{13}$$

then

$$E \equiv \lim_{r\to 1} E(r) = \frac{1}{2}\int_0^{\pi} \frac{d\alpha(\theta)}{\sin\theta}, \tag{14}$$

and is finite whenever this integral exists. However, since $F(z)$ is convex in the direction of the imaginary axis, and since $E(r)$ is the length of the segment of the real axis intercepted by the contour into which $|z| = r$ is mapped by $F(z)$, we have

$$|\mathcal{R}F(z_1) - \mathcal{R}F(z_2)| \leqq F(r) - F(-r) = E(r) \tag{15}$$

for all z_1 and z_2 on $|z| = r$. Thus $E(r)$ denotes the oscillation of the real part of $F(z)$ on $|z| = r$.

From (4) and (7) we have, by comparing coefficients on both sides of the equation (7),

$$a_n = \frac{1}{\pi}\int_0^{\pi} \frac{\sin n\theta}{\sin\theta}\,d\alpha(\theta), \tag{16}$$

$$|a_n| \leqq \frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin n\theta}{\sin\theta}\right| d\alpha(\theta) \leqq \frac{1}{\pi}\int_0^{\pi} \frac{d\alpha(\theta)}{\sin\theta}. \tag{17}$$

Whenever E is finite we have, by (14) and (17),

$$|a_n| \leqq \frac{2}{\pi}E, \qquad \text{for all } n, \tag{18}$$

$$\frac{1}{n+1}\sum_{k=1}^{n} |a_k| \leqq \frac{1}{\pi}\int_0^{\pi} M_n(\theta)\,\frac{d\alpha(\theta)}{\sin\theta}, \tag{19}$$

where

$$M_n(\theta) \equiv \frac{1}{n+1}\sum_{k=1}^{n} |\sin n\theta|.$$

However, as T. Gronwall has shown,*

* See T. Gronwall, Transactions of this Society, vol. 13 (1912), pp. 445–468.

$$M_n(\theta) < \sin z_0 = 0.72457\cdots, \tag{20}$$

where z_0 is the positive root of the equation $\tan(z_0/2) = z_0$. Further,

$$\lim_{n\to\infty} M_n(\theta) = M(\theta) \leqq \frac{2}{\pi}, \tag{21}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} |a_k| \leqq \frac{1}{\pi} \int_0^{\pi} \lim_{n\to\infty} M_n(\theta) \frac{d\alpha(\theta)}{\sin\theta}$$

$$\leqq \frac{2}{\pi} \frac{1}{\pi} \int_0^{\pi} \frac{d\alpha(\theta)}{\sin\theta} \leqq \left(\frac{2}{\pi}\right)^2 E. \tag{22}$$

Similarly,

$$\frac{1}{n+1} \sum_{k=1}^{n} |a_k| < \left(\frac{2\sin z_0}{\pi}\right) E < \left(\frac{1.45}{\pi}\right) E, \tag{23}$$

for all n.

Again, if we denote by Γ_n the expression

$$\Gamma_n = \max_{\theta} \sum_{k=1}^{n} \frac{|\sin k\theta|}{k}, \tag{24}$$

then we have*

$$\frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k} < \Gamma_n < \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k} + \frac{2}{\pi}, \tag{25}$$

$$\lim_{n\to\infty} \frac{\Gamma_n}{\log n} = \frac{2}{\pi}. \tag{26}$$

Hence by the method used above in (19) we may show that

$$\sum_{k=1}^{n} \frac{|a_k|}{k} < \left\{\left(\frac{2}{\pi}\right)^2 + \left(\frac{2}{\pi}\right)^2 \sum_{k=1}^{n} \frac{1}{k}\right\} E, \tag{27}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{\log n} \cdot \sum_{k=1}^{n} \frac{|a_k|}{k} \leqq \left(\frac{2}{\pi}\right)^2 E. \tag{28}$$

Let

* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, 1925, pp. 81 and 274.

$$A(n, \theta) \equiv \sum_{k=1}^{n} \frac{\sin k\theta}{k}. \tag{29}$$

Then the absolute maximum* of $A(n, \theta)$ is $A(n, \pi/(n+1))$. Consequently we obtain by the above method

$$\left| \sum_{k=1}^{n} \frac{a_k}{k} \right| \leqq \frac{2}{\pi} A\left(n, \frac{\pi}{n+1}\right) E. \tag{30}$$

For n odd the factor $2/\pi$ in (18) cannot be replaced by a smaller one, since for the function $f(z) = z(1+z^2)^{-1}$ we have

$$|a_{2n-1}| = 1, \quad a_{2n} = 0, \quad F(z) = \arctan z, \quad E = \frac{\pi}{2}.$$

Hence equality is attained by $z(1+z^2)^{-1}$ for every odd value of n. However, one cannot have equality for all n, even and odd, for a given function of the class under consideration, as this would contradict the inequality (22).

4. *A Class of Odd Typically-Real Functions.* Let I denote the class of odd functions

$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1} \tag{31}$$

with the properties

(a) $f(z)$ is regular for $|z| < 1$,

(b) $f(z)$ is real on the real axis, that is, b_{2n+1} is real for all n,

(c) $f(z)$ lies inside the jth quadrant whenever z is inside the jth quadrant for $|z| < 1$, $(j = 1, 2, 3, 4)$.

The class of odd functions regular and univalent for $|z| < 1$ and real on the real axis form a subclass of I.

THEOREM. *If*

$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}$$

belongs to class I, then

$$|b_3| \leqq 1, \quad |b_{2n-1}| + |b_{2n+1}| \leqq 2, \quad \sum_{k=1}^{n} \frac{b_{2k+1}}{k} \geqq -2.$$

* See G. Pólya and G. Szegö, loc. cit., p. 79.

PROOF. Let $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.

$$b_{2n+1}r^{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} v(r, \theta) \sin (2n + 1)\theta \, d\theta, \tag{32}$$

$$b_{2n+1}r^{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r, \theta) \cos (2n + 1)\theta \, d\theta. \tag{33}$$

On account of hypotheses (b) and (c) of the definition of class I we may write

$$b_{2n+1}r^{2n+1} = \frac{4}{\pi}\int_{0}^{\pi/2} v(r, \theta) \sin (2n + 1)\theta \, d\theta, \tag{34}$$

$$b_{2n+1}r^{2n+1} = \frac{4}{\pi}\int_{0}^{\pi/2} u(r, \theta) \cos (2n + 1)\theta \, d\theta, \tag{35}$$

where $v(r, \theta) > 0$, $u(r, \theta) > 0$ for $0 < \theta < \pi/2$, $r < 1$. From (34) we have

$$b_{2n+1}r^{2n+1} - b_{2n-1}r^{2n-1} = \frac{8}{\pi}\int_{0}^{\pi/2} v(r, \theta) \cos 2n\theta \sin \theta \, d\theta, \tag{36}$$

so that

$$\begin{aligned} \left| b_{2n+1}r^{2n+1} - b_{2n-1}r^{2n-1} \right| &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} \left| v(r, \theta) \sin \theta \right| d\theta \\ &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} v(r, \theta) \sin \theta \, d\theta \\ &\leqq 2r. \end{aligned}$$

Letting $r \to 1$, we have

$$\left| b_{2n-1} - b_{2n+1} \right| \leqq 2. \tag{37}$$

From (35) we have similarly

$$\begin{aligned} \left| b_{2n-1}r^{2n-1} + b_{2n+1}r^{2n+1} \right| &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} \left| u(r, \theta) \cos 2n\theta \cos \theta \right| d\theta \\ &\leqq \frac{8}{\pi}\int_{0}^{\pi/2} u(r, \theta) \cos \theta \, d\theta \\ &\leqq 2r. \end{aligned} \tag{38}$$

Letting $r \to 1$ again, we have

$$|b_{2n-1} + b_{2n+1}| \leqq 2. \tag{39}$$

Combining (37) and (39) we obtain

$$|b_{2n-1}| + |b_{2n+1}| \leqq 2, \qquad \text{(for all } n\text{)}. \tag{40}$$

The inequalities (40) were established by a different method for the subclass of I consisting of odd univalent functions real on the real axis by J. Dieudonné.* Further, since we have†

$$B(n, \theta) \equiv \sum_{k=1}^{n} \frac{\cos k\theta}{k} \geqq -1 \tag{41}$$

for all n, then

$$\sum_{k=1}^{n} \frac{(b_{2k+1}r^{2k+1} - b_{2k-1}r^{2k-1})}{k} = \frac{8}{\pi}\int_0^{\pi/2} B(n, \theta)v(r, \theta) \sin\theta \, d\theta \geqq -2r, \tag{42}$$

$$\sum_{k=1}^{n} \frac{(b_{2k+1} - b_{2k-1})}{k} \geqq -2, \tag{43}$$

and similarly

$$\sum_{k=1}^{n} \frac{(b_{2k+1} + b_{2k-1})}{k} \geqq -2. \tag{44}$$

On adding (43) and (44) we obtain also

$$\sum_{k=1}^{n} \frac{b_{2k+1}}{k} \geqq -2. \tag{45}$$

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* See J. Dieudonné, Annales de l'École Normale, vol. 48 (1931), p. 318.
† See G. Pólya und G. Szegö, loc. cit., p. 79.

DERIVATIVES OF HIGHER ORDER AS SINGLE LIMITS*

BY PHILIP FRANKLIN

1. *Introduction.* If a function has an nth derivative at a point, the value of this derivative may be calculated by taking the limit of a suitable quotient involving the nth difference, as the fundamental difference for the independent variable approaches zero. However, the limit of this quotient may exist without the function having the corresponding derivative. The principal result of this paper deals with the expression for unevenly spaced points, which reduces to the quotient mentioned above when the points are equidistant. It is shown that the existence of the limit of this expression, when the points close down in any way whatever to a fixed point, is a necessary and sufficient condition for the function to have an nth derivative throughout some neighborhood of the fixed point, continuous at the point. Some applications to finite Taylor developments are considered. A condition for the mere existence of the derivative is also given.

2. *Earlier Results.* A precise statement of the theorem on nth differences,† as proved by de la Vallée-Poussin, is that

$$\lim_{h\to 0} \frac{\Delta^n y}{h^n} = f^{(n)}(x_0), \tag{1}$$

if $y=f(x)$ has an nth derivative at x_0, where $\Delta^n y$ is the nth difference defined by the formulas

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x),$$

$$\Delta f(x) = f(x+h) - f(x).$$

The corresponding expression for unevenly spaced points is, except for a numerical factor, $n!$,

$$A_{n,k} = \sum_{j=1}^{n} \frac{f(x_j)}{(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_{n+1})}, \tag{2}$$

* Presented to the Society, December 27, 1933.

† de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, vol. 1, 1921, p. 73.

as is seen by comparing Newton's and Lagrange's interpolation formulas. It is sometimes convenient to write this in terms of determinants:

$$A_{n,k} = \frac{\left| f(x_j),\ x_j^{n-1},\ \cdots,\ x_j,\ 1 \right|}{\left| x_j^n,\ x_j^{n-1},\ \cdots,\ x_j,\ 1 \right|}. \tag{3}$$

We have written only the jth row, $(j=1, 2, \cdots, n+1)$, for the rest.

It has been shown by the writer* that if $y=f(x)$ has an nth derivative at a, continuous in some neighborhood of x_0, then as the points close down to a, that is,

$$\lim_{k\to\infty} \epsilon_k = 0, \quad \left| x_j - a \right| < \epsilon_k, \tag{4}$$

we shall have

$$f^{(n)}(a) = \lim_{k\to\infty} n! A_{n,k}. \tag{5}$$

A similar result was proved† for the case of a complex independent variable, with the points closing down in any way, and $f(x)$ an analytic function.

The limit using the nth difference is too broad to be used as a definition of a derivative. For this limit may exist without there being a derivative. An example is the function $y=bx$, when x is rational, $y=cx$, when x is irrational. If $a=0$, the points $0, h, 2h, \cdots, nh$, are either all rational or all irrational, so that at 0, $\Delta^2 y=0$, and $\lim \Delta^2 y/h^2=0$. However, the function fails to have a first derivative at the origin, and hence has no second derivative there.

Again, as a definition of mere existence of a derivative, the use of $A_{n,k}$ with points closing down arbitrarily is unsatisfactory since it fails to give correct results unless the derivative is continuous at the point a considered. For, if the n points are allowed to come together at a point near a, in the proper way, for example with equal spacing, they will give the derivative at this point. If the derivative is not continuous at a, this derivative will not approach the value at a, when the nearby point approaches a.

* Transactions of this Society, vol. 28 (1926), p. 403.

† This Bulletin, vol. 33 (1927), p. 465.

3. *Condition for a Continuous nth Derivative.* Consider a sequence of sets of $n+1$ points. Call the kth set S_k, and let its elements satisfy (4). Then we shall say that the set S_k closes down on a. For each set we may form $A_{n,k}$ by (2), or (3). We may now formulate* the following theorem.

THEOREM 1. *The existence of a unique limit for $A_{n,k}$ as the points of S_k close down on a in an arbitrary manner is a necessary and sufficient condition for $f(x)$ to have an nth derivative $f^n(a)$, continuous at a. The value of the derivative is given by* (5).

That the condition is necessary† follows from Rolle's theorem and Lagrange's interpolation formula. For

$$(6)\quad F(x) = \sum_{j=1}^{n} \frac{(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_{n+1})}{(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_{n+1})} f(x_j)$$

is a polynomial of the nth degree agreeing with $f(x)$ at the points of S_k. Hence, $f(x)-F(x)$ vanishes at $n+1$ points, and by successive application of Rolle's theorem its nth derivative vanishes at some point X, interior to the interval whose end points are the extreme points of S_k. That is,

$$(7)\qquad F^{(n)}(X) = f^{(n)}(X).$$

But from (2) and (6),

$$(8)\qquad F(x) = A_{n,k}x^n + P_{n-1}(x),$$

where $P_{n-1}(x)$ is a polynomial of the $(n-1)$th degree, so that

$$(9)\qquad F^{(n)}(X) = n!A_{n,k}.$$

Since the points of S_k are closing down on a, X approaches a as a limit, so that from the continuity of $f^{(n)}(x)$,

$$(10)\qquad \lim_{k\to\infty} n!A_{n,k} = \lim_{X\to a} f^{(n)}(X) = f^{(n)}(a).$$

This proves the necessity of the condition.

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* Hassler Whitney, Transactions of this Society, vol. 36 (1934), p. 369. This paper contains certain similar results of this type, involving uniformity conditions.

† Transactions of this Society, vol. 28 (1926), p. 403.

We shall prove the sufficiency of the condition by mathematical induction. For $n=1$, we have

$$A_{1,k} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{11}$$

By taking $x_1 = a$, and letting x_2 approach a, we see that $f'(a)$ exists. Again, since the limit must exist when x_2 approaches x_1, and then if we let x_2 remain coincident with x_1 and approach a, we have

$$f(a) = \lim_{x_1 \to a} \lim_{x_2 \to x_1} A_{1,k} = \lim_{x_1 \to a} f'(x_1), \tag{12}$$

so that $f'(x_1)$ exists for x_1 near a, and is continuous at a. We accordingly assume as the basis of the induction that the condition is sufficient for $n = 1, 2, \cdots, N$. Now consider $A_{n,k}$, and to simplify the notation change the origin so that $a = 0$. Put $x_1 = x$, $x_2 = 0$, and let x approach zero. Then all the terms in the sum (2) for $A_{n,k}$ except the first two approach limits, and the first two may be written:

$$\frac{f(x) - f(0)}{x} \frac{1}{(x - x_3) \cdots (x - x_{n+1})} + \frac{f(0)}{x}\left[\frac{1}{(x - x_3) \cdots (x - x_{n+1})} - \frac{1}{(-x_3) \cdots (-x_{n+1})}\right].$$

As the second of these approaches a limit for x approaching zero, and the product does, we have shown the existence of

$$f'(0) = \lim \frac{f(x) - f(0)}{x}. \tag{13}$$

We now define a continuous function $v(x)$ by the equations

$$f(x) = f(0) + xv(x), \qquad (x \neq 0), \tag{14}$$

$$v(0) = f'(0). \tag{15}$$

If we calculate $A_{n,k}$ by (3), for $x_{n+1} = 0$, and make use of (14), we find, after reduction,

$$A_{n,k} = \frac{\left| v(x_j), x_j^{n-2}, \cdots, x_j, 1 \right|}{\left| x_j^{n-1}, x_j^{n-2}, \cdots, x_j, 1 \right|}, \qquad (j = 1, 2, \cdots, n) \tag{16}$$

As this has the same form as (3), with $f(x)$ replaced by $v(x)$, and n by $n-1$, by the basis of our induction, it follows that

$$\lim N!A_{N+1,k} = v^{(N)}(0), \tag{17}$$

and that $v^{(N)}(x)$ exists in some neighborhood of $a=0$, and is continuous at this point. The derivatives of $v(x)$ of lower order have this same property.

By differentiating (14), we find

$$f^{(n)}(x) = xv^{(n)}(x) + nv^{(n-1)}(x). \tag{18}$$

This is in the first instance only established for $x \neq 0$, but follows for $x=0$ from the continuity* of the derivatives of $v(x)$, which shows that for $n \leqq N$, $f^{(n)}(0)$ exists and is given by

$$f^{(n)}(0) = nv^{(n-1)}(0). \tag{19}$$

Thus

$$\frac{f^{(N)}(x) - f^{(N)}(0)}{x} = v^{(N)}(x) + N\frac{v^{(N-1)}(x) - v^{(N-1)}(0)}{x}, \tag{20}$$

and on letting x approach zero, we show the existence of

$$f^{(N+1)}(0) = (N+1)v^{N}(0). \tag{21}$$

Comparison with (17) shows that

$$\lim (N+1)!A_{N+1,k} = f^{(N+1)}(0) \tag{22}$$

for the special approach used, and hence for any approach since the limit is assumed to be unique. In particular, if we first let the $(N+1)$ points of S_k coalesce at x_1 near $a=0$, and then remaining coincident with x_1 approach $a=0$, we shall have

$$f^{(N+1)}(0) = \lim_{x_1 \to 0} f^{(N+1)}(x_1),$$

since the point x_1 may be taken as the 0 in (22), when sufficiently near a for the limit to exist. This establishes the continuity of $f^{(N+1)}(x)$ at 0, and so proves the sufficiency of the condition.

4. *Taylor Developments.* If a function has an nth derivative at a point, $f^{(n)}(a)$, and we form its Taylor development

* de la Vallée-Poussin, loc. cit., p. 127, number 5.

$$f(a+h) = f(a) + \frac{h}{1} f'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} M(h); \tag{23}$$

then, as shown by W. H. Young,*

$$\lim_{h \to 0} M(h) = f^{(n)}(a). \tag{24}$$

We may have a development of this form

$$f(a+h) = A_0 + \frac{h}{1} A_1 + \cdots + \frac{h^{n-1}}{(n-1)!} A_{n-1} + \frac{h^n}{n!} M(h), \tag{25}$$

with

$$\lim_{h \to 0} M(h) = M(0), \tag{26}$$

without $f(x)$ having an nth derivative at $x = a$.

In fact, such a development implies nothing as to the derivatives of $f(x)$ beyond the fact that $f'(a) = A_1$. This is shown by the following example:

$$f(x) = x^{n+1} \sin x^{-n-1}, \qquad x \neq 0, \qquad f(0) = 0.$$

There is a development about the origin, of the form (25), with

$$A_1 = A_2 = \cdots = M(0) = 0, \qquad M(h) = n!h \sin h^{-n-1}.$$

However $f'(x)$ is discontinuous at 0, so that there is no second derivative.

We give a condition for a development of type (25) in the following theorem.

THEOREM 2. *The existence of the limit $A_{n,k}$ for all sets of points closing down to a in such a way that a bound K exists such that*

$$|x_j - a| \leqq K |x_j - x_i|, \qquad (i \neq j), \tag{27}$$

is a necessary and sufficient condition for $f(x)$ to have a Taylor development of the nth order about the point a.

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* W. H. Young, *The Fundamental Theorems of the Differential Calculus*, Cambridge Tract, No. 10 (1910), p. 16; see de la Vallée Poussin, loc. cit., p. 77.

It may be noted that the condition (27) prevents the points from having mutual distances small compared with their distances from a. In particular, two points can not coalesce unless they both coincide with a.

To prove the necessity, we simplify the notation by taking $a=0$, so that our assumed development is

$$f(x) = A_0 + \frac{x}{1}A_1 + \cdots + \frac{x^{n-1}}{(n-1)!}A_{n-1} + \frac{x^n}{n!}M(x). \tag{28}$$

Now calculate $A_{n,k}$ by using (3) and (28) for the functions $f(x_j)$ in the numerator. The result is

$$n!A_{n,k} = \frac{\left| x_j^n M(x_j), x_j^{n-1}, \cdots, x_j, 1 \right|}{\left| x_j^n, x_j^{n-1}, \cdots, x_j, 1 \right|} = M(0) + \sum_{j=1}^{n+1} \frac{x_j^n [M(x_j) - M(0)]}{(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_{n+1})}. \tag{29}$$

From (27) we have $|x_j| \leqq K|x_j - x_i|$, $(i \neq j)$, so that each term in the summation of (29) is dominated by $K^n|M(x_j) - M(0)|$ and approaches zero when $M(x_j) \to M(0)$. If any point x_j reaches $a=0$, the corresponding term does not appear in the summation. Thus, if there is a development (28), when (27) holds,

$$\lim n!A_{n,k} = M(0). \tag{30}$$

For the sufficiency, we use the reasoning previously given to establish (13), which is valid since the conditions there used are compatible with (27). We define $v(x)$ by (14) and (15), and consider (16). Now use mathematical induction. For $n=1$, the existence of a derivative implies the existence of a development (28). Assume the theorem holds for $n=1, 2, \cdots, N-1$. Then from this, applied to $v(x)$, we have

$$\frac{f(x) - f(0)}{x} = v(x) = B_0 + B_1 x + \cdots + \frac{x^{N-1}}{(N-1)!}B(x), \tag{31}$$

and from this

$$f(x) = f(0) + B_0 x + B_1 x^2 + \cdots + \frac{x^N}{N!}NB(x), \tag{32}$$

which proves the result for $n = N$, completing the induction.

COROLLARY. *If the function $f(x)$ has an nth derivative at a, then as S_k closes down to a, subject to* (27),

$$\lim n! A_{n,k} = f^{n}(a). \tag{33}$$

For, under the hypothesis, there is a development given by (23) and (24), and the result follows from (30).

5. *Contact.* The order of contact of two curves is usually stated in terms of derivatives. However, as the derivatives are merely used to get the Taylor development, if we have a Taylor development at hand it may be used to define order of contact. Thus, the example previously mentioned,

$$y = x^{n+1} \sin x^{-n-1}, \quad x \neq 0; \qquad y = 0, \quad x = 0, \tag{34}$$

is a curve having contact of the nth order with the x axis at the origin.

Similarly, certain other geometric concepts may be reduced to that of contact. Thus, in this sense, the curve

$$y = 2kx^2 + x^3 \sin \frac{1}{x} \tag{35}$$

has curvature k at the origin. That is, it has second order contact with a circle of radius $1/k$, the circle of curvature. There is no second derivative at the origin. However, by Theorem 2, if a circle is passed through three points on the curve, which close down to the origin subject to the condition (27), it will approach the circle of curvature.

In a space where distances, rather than coordinates, are given, it would be natural to replace (27) by a similar condition on the distances. This would lead to an intrinsic definition of curvature more general than that of Menger,* based on the expression for the radius of the circle through the three points, allowed to close down without restriction.

6. *Condition for an nth Derivative.* We shall now discuss a condition for the mere existence of a derivative.

THEOREM 3. *The existence of a unique limit for $A_{n,k}$ as one of the points of S_k remains at a, while the rest close down on a, is a necessary and sufficient condition for the existence of an nth derivative, $f^{(n)}(a)$.*

* Karl Menger, Mathematische Annalen, vol. 103 (1930), p. 480.

With the restriction stated, by the reasoning previously used for (13) and (16), we see that an alternative statement of the condition for the existence of $f^{(n)}(a)$ can be given in terms of the existence and continuity of $v^{(n-1)}(a)$, where

$$v(x) = \frac{f(x) - f(0)}{x}. \tag{36}$$

That the continuity of $v^{(n-1)}(a)$ insures the existence of $f^{(n)}(x)$ follows from (18) and from (20) with $N = n-1$.

To show the converse, we note that $v(0) = f'(0)$, and assume that if $f^{(n+1)}(0)$ exists,

$$v^{(n)}(0) = \frac{f^{(n+1)}(0)}{n+1}, \tag{37}$$

as the basis of an induction. We deduce from (36) that

$$\begin{aligned} v^{(n)}(x) = f^{(n)}(x)x^{-1} - nf^{(n-1)}(x)x^{-2} + \cdots \\ + (-1)^n n![f(x) - f(0)]x^{-n-1}. \end{aligned} \tag{38}$$

Thus

$$\begin{aligned} \frac{v^{(n)}(x) - v^{(n)}(0)}{x} = \frac{f^{(n)}(x)x^n - nf^{(n-1)}(x)x^{n-1}}{x^{n+2}} + \cdots \\ + \frac{(-1)^n n![f(x) - f(0)] - x^{n+1}f^{(n+1)}(0)/(n+1)}{x^{n+2}}. \end{aligned}$$

By L'Hospital's rule, the application of which telescopes the terms, this will approach a limit if

$$\frac{f^{(n+1)}(x)x^n - f^{(n+1)}(0)x^n}{(n+2)x^{n+1}} = \frac{f^{(n+1)}(x) - f^{(n+1)}(0)}{(n+2)x} \tag{39}$$

does. As the latter expression approaches $f^{(n+2)}(0)/(n+2)$, if this derivative exists, we see that in this case

$$v^{(n+1)}(0) = \frac{f^{(n+2)}(0)}{(n+2)}, \tag{40}$$

which completes our induction.

This proves that if $f^{(n+1)}(0)$ exists, $v^{(n)}(0)$ exists. In this case, in some neighborhood of 0, $f^{(n)}(x)$ exists and is continuous. Thus,

from (38), $v^{(n)}(x)$ exists, and is continuous except perhaps at $x=0$. To settle this point, we write

$$v^{(n)}(x) - v^{(n)}(0) = f^{(n)}(x)x^{-1} - f^{(n)}(0)x^{-1} + f^{(n)}(0)x^{-1}$$
$$- nf^{(n-1)}(x)x^{-2} + \cdots + (-1)^n n![f(x) - f(0)]x^{-n-1} \tag{41}$$
$$- f^{(n+1)}(0)/(n+1).$$

The limit of the first two terms is $f^{(n+1)}(0)$. The remaining terms may be written with denominator x^{n+1}, and their limit found by L'Hospital's rule as for (39). Here we find the limit is that of

$$\frac{nf^{(n)}(0)x^{n-1} - nf^{(n)}(x)x^{n-1} - f^{(n+1)}(0)x^n}{(n+1)x^n},$$

which is

$$\lim\left[-\frac{n}{n+1}\frac{f^{(n)}(x) - f^{(n)}(0)}{x} - \frac{f^{(n-1)}(0)}{n+1}\right] = -f^{(n+1)}(0). \tag{42}$$

Thus, from (41),

$$\lim\,[v^{(n)}(x) - v^{(n)}(0)] = 0,$$

and the continuity of $v^{(n)}(x)$ for $x=0$ is established. This completes the proof of Theorem 2.

As a simple example of the theorem, we have the function

$$f(x) = x^4 \sin\frac{1}{x}, \quad x \neq 0, \qquad f(0) = 0,$$

which has a second derivative, discontinuous for $x=0$. The corresponding $v(x)$ is

$$v(x) = x^3 \sin\frac{1}{x}, \quad x \neq 0, \qquad v(0) = 0,$$

which has a continuous first derivative.

We formulate in the following corollary the alternative condition used above.

COROLLARY. *A necessary and sufficient condition for the existence of an nth derivative, $f^{(n)}(a)$, is the existence of an $(n-1)$th derivative, $v^{(n-1)}(x)$, continuous in some neighborhood of a, where $v(x)$ is related to $f(x)$ by* (36).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

A CONVERGENCE FACTOR THEOREM IN THE THEORY OF SUMMABLE SERIES*

BY H. L. GARABEDIAN

1. *Introduction.* The object of this paper is to write down sufficient conditions to ensure that any definition of summability of the *convergence factor type*† be more effective than or *include*‡ the definition of de la Vallée-Poussin.

A series

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots \tag{1}$$

is said to be summable by the method of de la Vallée-Poussin (or summable *VP*) to the sum S if $\lim_{n\to\infty} v_n = S$, where

$$\begin{aligned} v_n &= u_0 + \sum_{k=1}^{n} \frac{C_{2n,\,n-k}}{C_{2n,\,n}} u_k \\ &= u_0 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{(n+1)(n+2)\cdots(n+k)} u_k. \end{aligned} \tag{2}$$

For the sake of convenience, we shall define the series (1) to be ϕ-summable to the sum S provided that the set of functions, $\phi_n(\alpha)$, $(n=0, 1, 2, \cdots)$, are defined for a set of values $E(\alpha)$ having at least one limit point α_0, not of the set; the series $\sum_{n=0}^{\infty}\phi_n(\alpha)u_n$ converges over $E(\alpha)$; and

$$\lim_{\alpha\to\alpha_0} \sum_{n=0}^{\infty} \phi_n(\alpha)u_n = S.$$

The theorem studied in this paper is stated as follows.

THEOREM. *If* (i) *the series* (1) *is summable VP to the sum* S, (ii) *the set of functions,* $\phi_n(\alpha)$, $(n=0, 1, 2, \cdots)$, *defined for a set of values* $E(\alpha)$ *having at least one limit point* α_0, *not of the set, satisfies the conditions*

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* Presented to the Society, September 5, 1934.

† C. N. Moore, Transactions of this Society, vol. 8 (1907), p. 299.

‡ W. A. Hurwitz, this Bulletin, vol. 28 (1922), p. 17.

$$(a_1) \quad \lim_{\alpha \to \alpha_0} \phi_n(\alpha) = 1, \qquad (a_2) \quad \lim_{n \to \infty} \nu^n \phi_n(\alpha) = 0, \text{ over } E(\alpha),$$

$$(a_3) \quad \sum_{n=0}^{\infty} C_{2n,\,n} \left| \sum_{\lambda=0}^{\infty} C_{\lambda+2n,\,2n} \Delta^{2n+2} \phi_{2\lambda+n}(\alpha) \right| < K, \text{ over } E(\alpha),$$

where ν and K are independent of α and n, and $\nu > e^k$, (iii) ϕ-summability includes Cesàro summability of arbitrary integral order; then the series $\sum_{n=0}^{\infty} \phi_n(\alpha) u_n$ converges over $E(\alpha)$, and*

$$\lim_{\alpha \to \alpha_0} \sum_{n=0}^{\infty} \phi_n(\alpha) u_n = S.$$

We conclude this introduction with the remark that the *raison d'être* of this theorem may be traced to a somewhat similar theorem, due to Bromwich,† in which sufficient conditions are given in order that ϕ-summability include (Ck) summability. It will be observed that an analogy can be drawn between the conditions of the present theorem and those of the Bromwich theorem despite the discrepancy in complexity of the problems involved.

2. *Proof of Theorem.* At the outset we shall consider the de la Vallée-Poussin matrix and its inverse, denoted respectively by (VP) and $(VP)^{-1}$, and their use in satisfying some of the exigencies of the problem at hand.

The matrix (VP) is of triangular type, that is,

$$(VP) = \begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{2} & 0 & 0 & \cdots \\ 1 & \frac{2}{3} & \frac{1}{6} & 0 & \cdots \\ 1 & \frac{3}{4} & \frac{3}{10} & \frac{1}{20} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}.$$

* Let us recall that VP summability includes (Ck) summability, that is, Cesàro summability of order k. This result was obtained independently and virtually simultaneously by T. H. Gronwall, Comptes Rendus, vol. 158 (1914), p. 1664, and C. N. Moore, Comptes Rendus, vol. 158 (1914), p. 1774.

† Mathematische Annalen, vol. 65 (1907–08), p. 351.

The inverse matrix $(VP)^{-1}$ is known to be*

$$(VP)^{-1} = \begin{Vmatrix} b_{00} & 0 & 0 & 0 & \cdots \\ b_{10} & b_{11} & 0 & 0 & \cdots \\ b_{20} & b_{21} & b_{22} & 0 & \cdots \\ b_{30} & b_{31} & b_{32} & b_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

$$= \begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & 2 & 0 & 0 & \cdots \\ 2 & -8 & 6 & 0 & \cdots \\ -2 & 18 & -36 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}.$$

The matrix $(VP)^{-1}$ gives the solution of the equations (2) for the u's in terms of the v's:

$$(3) \quad u_0 = v_0, \quad u_n = 2\sum_{k=0}^{n} (-1)^{n-k} C_{n,k}\, C_{n+k-1,k}\, v_k,$$

$$(n = 1, 2, 3, \cdots).$$

Note that

$$b_{00} = 1, \; b_{nk} = 2(-1)^{n-k} C_{n,k}\, C_{n+k-1,k},$$

$$(k = 0, 1, 2, \cdots, n; \; n \neq 0).$$

Form the series

$$(4) \qquad \sum_{n=0}^{\infty} \phi_n(\alpha) u_n,$$

and replace u_n in (4) by its value in (3) to get

$$(5) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{nk} \phi_n(\alpha) v_k.$$

At this juncture, we find it expedient to pause temporarily to prove the following lemma.

* George Rutledge, Journal of Mathematics and Physics, M. I. T., vol. 11 (1932), p. 76.

LEMMA 1.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{nk}\phi_n(\alpha)v_k = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} b_{kn}\phi_k(\alpha)v_n. \tag{6}$$

The right-hand member of (6) is obtained from (5) by an interchange in the order of limits. In order to prove this identity we must establish completely the validity of the infinite processes involved.

We find it convenient to write out one of the members of (6) in the form of a double series:

$$\begin{array}{l} b_{00}\phi_0v_0+\ 0\ +\ 0\ +\ 0\ +\ 0\ +\ \cdots \\ +b_{10}\phi_1v_0+b_{11}\phi_1v_1+\ 0\ +\ 0\ +\ 0\ +\ \cdots \\ +b_{20}\phi_2v_0+b_{21}\phi_2v_1+b_{22}\phi_2v_2+\ 0\ +\ 0\ +\ \cdots \\ +b_{30}\phi_3v_0+b_{31}\phi_3v_1+b_{32}\phi_3v_2+b_{33}\phi_3v_3+\ 0\ +\ \cdots \\ +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{array} \tag{7}$$

The left member of equation (6) represents the double series (7) *summed by rows*, and the right-hand member of (6) represents the series (7) *summed by columns*. We wish to show that the sums obtained in these two different fashions are identical. Now, it follows from a theorem due to Pringsheim* that if the rows and columns of a double series converge, and if the double series is convergent, then the interchange of limits implied in (6) is permissible. We shall first show that the individual rows and the individual columns of the series (7) converge, and secondly execute the more difficult task of showing that the double series itself converges.

The individual rows of (7) converge since the terms are all zero from a certain point on. The individual columns are series of the form $\sum_{k=n}^{\infty}\phi_k(\alpha)b_{kn}v_n$, $(n=0, 1, 2, \cdots)$, or

$$\begin{cases} 2\sum_{k=n}^{\infty}(-1)^{k-n}C_{k,n}C_{k+n-1,n}\phi_k(\alpha)v_n, \\ \qquad\qquad (n = 1, 2, \cdots), \\ \left[\phi_0 + 2\sum_{k=1}^{\infty}(-1)^k\phi_k(\alpha)\right]v_0. \end{cases} \tag{8}$$

* Bromwich, *Theory of Infinite Series*, 1st or 2d edition, §30.

We have $b_{kn}=O[(k+n)!/(n!)^2(k-n)!]$. Using Stirling's well known formula, for the Γ-function, we can write

$$b_{kn} = O\left[(k+n)^{k+n}/\left\{n^{2n}(k-n)^{k-n}\right\}\right]. \tag{9}$$

Now, set $k=n+m$, where m is a positive integer, and we obtain

$$b_{kn} = O\left[(2n+m)^{2n+m}/\left\{n^{2n}m^m\right\}\right] = O[4^n e^m e^{2n}].$$

Substituting $m=k-n$ in this last expression we can write

$$b_{kn}\phi_k = O[4^n e^{k+n}\phi_k] = O[e^{k+3n}\phi_k] = O[e^{4k}\phi_k]. \tag{10}$$

From condition (a_2) of the hypotheses we have $\phi_k=O(\nu^{-k})$, where $\nu>e^5$. Consequently, it follows from (10) that

$$b_{kn}\phi_k = O(e^{-k}). \tag{11}$$

The equation (11) implies the convergence of the series (8) and hence the convergence of the individual columns.

Let us now form the sum, s_{pq}, of the pq terms taken from the first p rows and the first q columns of the series (7). The convergence of s_{pq} as p and q tend to infinity in such a manner that $p \leqq q$ follows from the existence of the left-hand side of (6). Note that the existence of this expression is in turn implied by the existence of the series (4). To establish the convergence of the series (4) we recall first of all that, since $\lim_{k\to\infty} v_k$ exists, v_k is bounded. Then, from (9) and the discussion above it follows that $b_{nk}v_k=O(e^{n+3k})$, $u_n=\sum_{k=0}^{n} b_{nk}v_k=O(e^{4n})$ and $\phi_n u_n=O(e^{-n})$. This suffices to prove the convergence of the series (4). For the case $p>q$ we must show that the additional terms included are negligible. This additional block of terms is

$$\begin{aligned} &b_{q,0}\phi_q v_0 + b_{q,1}\phi_q v_1 + \cdots + b_{q,q}\phi_q v_q \\ &+ b_{q+1,0}\phi_{q+1}v_0 + b_{q+1,1}\phi_{q+1}v_1 + \cdots + b_{q+1,q}\phi_{q+1}v_q \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ b_{p,0}\phi_p v_0 + b_{p,1}\phi_p v_1 + \cdots + b_{p,q}\phi_p v_q. \end{aligned} \tag{12}$$

Designate an arbitrary one of these terms by $b_{i,j}\phi_i v_j$,

$$i = q, q+1, \cdots, p; \quad j = 0, 1, \cdots, q; \quad p \geqq i \geqq q; \quad q \geqq j \geqq 0.$$

We have

$$b_{i,j}\phi_i v_j = 2(-1)^{i-j} C_{i,j} C_{i+j-1,j}\phi_i v_j.$$

From (11) and the boundedness of v_j it follows that

$$b_{ij}\phi_i v_j = O(e^{-i}). \tag{13}$$

The equation (13) implies that the sum of the terms $b_{ij}\phi_i v_j$ of the expression (12) can be made small enough to be negligible by taking q sufficiently large. In turn, this implies the convergence of s_{pq}. Accordingly, we have established the validity of the identity (6).

Now, to return to our main argument, it follows from (4), (5), and the lemma above that we can set $F(\alpha)=\sum_{n=0}^{\infty}\phi_n(\alpha)u_n$ and write

$$F(\alpha) = \sum_{n=0}^{\infty} c_{\alpha n} v_n, \qquad c_{\alpha n} = \sum_{k=n}^{\infty} \phi_k(\alpha) b_{kn}. \tag{14}$$

Since $\lim_{n\to\infty} v_n = S$, it remains to show that the method of summation, of the convergence factor type, defined by (14) is *regular*,* which is to say that $\lim_{\alpha\to\alpha_0} F(\alpha) = S$. Accordingly, it is required in the present case that

$$\lim_{\alpha\to\alpha_0} c_{\alpha n} = 0 \text{ for every } n, \tag{b_1}$$

$$\lim_{\alpha\to\alpha_0} \sum_{n=0}^{\infty} c_{\alpha n} = 1, \tag{b_2}$$

$$\sum_{n=0}^{\infty} |c_{\alpha n}| < K \text{ over } E(\alpha), \tag{b_3}$$

K independent of α.

It remains to prove that the conditions (b) are fulfilled whenever the conditions of the theorem are satisfied. Recall that

$$c_{\alpha 0} = \phi_0 + 2\sum_{k=1}^{\infty} (-1)^k \phi_k(\alpha), \tag{15}$$

and, for $n=1, 2, \cdots$,

$$c_{\alpha n} = 2\sum_{k=n}^{\infty} (-1)^{k-n} C_{k,n}\, C_{k+n-1,n}\, \phi_k(\alpha), \tag{16}$$

and notice that, since $b_{kn}=0$, $(k<n)$,

$$c_{\alpha n} = \sum_{k=n}^{\infty} \phi_k(\alpha) b_{kn} = \sum_{k=0}^{\infty} \phi_k(\alpha) b_{kn}. \tag{17}$$

* Hurwitz, loc. cit., p. 20.

At this point, the following identity* proves to be of fundamental importance:

$$\sum_{k=0}^{\infty} b_{kn}x^k = (1-x)^{2n+2}\sum_{k=0}^{\infty} S_k^{(2n+1)}x^k, \quad (|x|<1), \tag{18}$$

where

$$S_k^{(2n+1)} = C_{2n+k+1,\,2n+1}\,b_{0n} + C_{2n+k,\,2n+1}\,b_{1n} + C_{2n+k-1,\,2n+1}\,b_{2n} + \cdots + C_{2n+1,\,2n+1}\,b_{kn}.$$

It results at once from (18) that

$$b_{kn} = S_k^{(2n+1)} - C_{2n+2,1}\,S_{k-1}^{(2n+1)} + C_{2n+2,2}\,S_{k-2}^{(2n+1)} - \cdots + (-1)^{2n+2}C_{2n+2,\,2n+2}S_{k-2n-2}^{(2n+1)}, \tag{19}$$

where it is understood that when a negative subscript occurs in the formula the corresponding $S^{(2n+1)}$ is to be replaced by zero.

Substitute the value for b_{kn} in (19) in the equation (17) to get

$$c_{an} = \sum_{k=0}^{\infty}\left[S_k^{(2n+1)} - C_{2n+2,1}\,S_{k-1}^{(2n+1)} + C_{2n+2,2}\,S_{k-2}^{(2n+1)} - \cdots + (-1)^{2n+2}C_{2n+2,\,2n+2}S_{k-2n-2}^{(2n+1)}\right]\phi_k(\alpha). \tag{20}$$

Ordering the terms of (20) with respect to $S_k^{(2n+1)}$, we obtain formally

$$c_{an} = \sum_{k=0}^{\infty} S_k^{(2n+1)}\left[\phi_k(\alpha) - C_{2n+2,1}\,\phi_{k+1}(\alpha) + \cdots + (-1)^r C_{2n+2,\,r}\,\phi_{k+r}(\alpha) + \cdots + (-1)^{2n+2}\phi_{k+2n+2}(\alpha)\right],$$

or

$$c_{an} = \sum_{k=n}^{\infty} S_k^{(2n+1)}\Delta^{2n+2}\phi_k(\alpha), \tag{21}$$

where we notice that $S_k^{(2n+1)} = 0$, $(k<n)$. In order to justify this

* Bromwich, loc. cit., 1st ed., p. 311.

last step it will suffice to establish the absolute convergence of the series

$$\sum_{k=i}^{\infty} S_{k-i}^{(2n+1)} \phi_k(\alpha), \qquad (i = 0, 1, 2, \cdots, 2n+2). \tag{22}$$

To this end and for further reference we delay once again to prove a fundamental lemma.

LEMMA 2.

$$S_k^{(2n+1)} = C_{2n,n}\, C_{\lambda+2n,2n}, \qquad (k = 2\lambda + n;\ n = 1, 2, \cdots). \tag{23}$$

From equation (18) we have the formula

$$\sum_{k=0}^{\infty} S_k^{(2n+1)} x^k = 2(1-x)^{-(2n+2)} \sum_{k=0}^{\infty} (-1)^{k-n}\, C_{k,n}\, C_{k+n-1,n}\, x^k, \tag{24}$$

which is needed later in the proof.

Multiplying the binomial expansion of $(1+x)^{-(2n+1)}$ by $C_{2n,n}x^n(1-x)$, we obtain

$$C_{2n,n} \frac{x^n(1-x)}{(1+x)^{2n+1}} = \frac{2}{(n!)^2} \sum_{j=0}^{\infty} (-1)^j (n+j)(2n+j-1)\cdots(j+1)x^{j+n}. \tag{25}$$

Multiplying both sides of (25) by $(1-x)^{-(2n+2)}$ and setting $j+n=k$ in the right member, we get

$$C_{2n,n}\, x^n(1-x^2)^{-(2n+1)} = 2(1-x)^{-(2n+2)} \sum_{k=0}^{\infty} (-1)^{k-n}\, C_{k,n}\, C_{n+k-1,n}\, x^k. \tag{26}$$

Now, eliminating the expression on the right-hand side of (26) between formulas (23) and (26), we have

$$\sum_{k=0}^{\infty} S_k^{(2n+1)} x^k = C_{2n,n}\, x^n (1 - x^2)^{-(2n+1)}$$
$$= C_{2n,n} \sum_{\lambda=0}^{\infty} C_{\lambda+2n,2n}\, x^{2\lambda+n}.$$

Our lemma follows at once from this identity.

From (22) and (23) we have

$$\sum_{k=i}^{\infty} S_{k-i}^{(2n+1)} \phi_k(\alpha) = \sum_{m=0}^{\infty} S_m^{(2n+1)} \phi_{m+i}(\alpha)$$
$$= C_{2n,n} \sum_{\lambda=0}^{\infty} C_{\lambda+2n,2n}\, \phi_{2\lambda+n+i}(\alpha).$$

Moreover, it is easy to prove the formulas $C_{\lambda+2n,2n} = O(e^{\lambda})$ and $\phi_{2\lambda+n+i}(\alpha) = O[e^{-(2\lambda+i)}]$. Consequently,

$$C_{\lambda+2n,2n}\phi_{2\lambda+n+i}(\alpha) = O[e^{-(\lambda+i)}],$$

and the absolute convergence of the series (22) is assured.

Now, formulas (21) and (23) are used to eliminate $S_k^{(2n+1)}$ and obtain

$$(27) \quad c_{\alpha n} = C_{2n,n} \sum_{\lambda=0}^{\infty} C_{\lambda+2n,2n}\, \Delta^{2n+2}\phi_{2\lambda+n}(\alpha), \qquad (n = 1, 2, \cdots).$$

Furthermore, it is easily proved from (15) that

$$(28) \quad c_{\alpha 0} = -\phi_0(\alpha) + 2\sum_{k=0}^{\infty} (-1)^k \phi_k(\alpha) = \sum_{\lambda=0}^{\infty} \Delta^2 \phi_{2\lambda}(\alpha).$$

Accordingly, formula (27) also holds for $n = 0$.

We are now prepared to discuss the conditions (b), first of all (b_1). Since the series $\sum_{k=0}^{\infty}(-1)^k$ is summable ($C1$) to the value $1/2$, and since, by condition (iii) of the hypotheses, ϕ-summability includes ($C1$) summability, it follows from (28) that $\lim_{\alpha\to\alpha_0} c_{\alpha 0} = 0$. Consequently, the condition (b_1) is satisfied for $n = 0$. Let us now focus our attention on the series

$$\sum_{k=n}^{\infty} C_{k,n}\, C_{k+n-1,n}(-1)^{k-n}, \qquad (n = 1, 2, \cdots).$$

If we form the Cesàro mean of order $2n+1$ for this series and use formula (23), we have

$$\lim_{k\to\infty} S_k^{(2n+1)}/C_{k+2n+1,2n+1}$$

$$= \lim_{\lambda\to\infty} C_{2n,n}\, C_{\lambda+2n,2n}/C_{\lambda+4n+1,2n+1} = 0, \qquad (n = 1, 2, \cdots).$$

Accordingly, the series in question is summable $(C, 2n+1)$ to the value zero. It follows a fortiori from formula (16) that $\lim_{\alpha\to\alpha_0} c_{an}=0$, $(n=1, 2, \cdots)$. This completes the proof that condition (b_1) is satisfied for all values of n.

Let us consider condition (b_2). We find from formula (17) that

$$\sum_{n=0}^{\infty} c_{an} = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} \phi_k(\alpha) b_{kn}. \tag{29}$$

An interchange in the order of summation in the double series (29) gives

$$\sum_{n=0}^{\infty} c_{an} = \sum_{n=0}^{\infty} \phi_n(\alpha) \sum_{k=0}^{n} b_{nk}. \tag{30}$$

This operation is justified by the identity obtained by equating the v's to unity in Lemma 1. It is quite evident from the matrix $(VP)^{-1}$ that $\sum_{k=0}^{n} b_{nk}=0$, $(n=1, 2, \cdots)$, and this can readily be proved by complete induction. Moreover, $b_{00}=1$. It follows from equation (30) that $\sum_{n=0}^{\infty} c_{an}=\phi_0(\alpha)$, and hence that $\lim_{\alpha\to\alpha_0}\sum_{n=0}^{\infty} c_{an}=1$. Accordingly, condition (b_2) has been shown to be satisfied.

Finally, it follows from (27) and the subsequent discussion that

$$\sum_{n=0}^{\infty} |c_{an}| = \sum_{n=0}^{\infty} C_{2n,n} \left| \sum_{\lambda=0}^{\infty} C_{\lambda+2n,2n} \Delta^{2n+2} \phi_{2\lambda+n}(\alpha) \right|,$$

and this expression is clearly uniformly bounded, over $E(\alpha)$, provided condition (a_3) obtains. Thus, the last of the conditions (b) has been shown to be satisfied. This completes the proof of our theorem.

NORTHWESTERN UNIVERSITY

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

ANN ARBOR, MICHIGAN, SUMMER MEETING AND COLLOQUIUM, September 10-14, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than August 6. Abstracts received by July 6 will appear in the July issue of the BULLETIN. Professor H. S. Vandiver will deliver a series of colloquium lectures on *Fermat's last theorem and related topics in number theory*. By invitation of the Program Committee, Professor G. Y. Rainich will deliver an address entitled *Product integrals*, and Professor G. T. Whyburn an address entitled *On the structure of continua*.

NEW YORK CITY, October 26, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, Low Memorial Library, Columbia University, New York City, not later than September 28. Abstracts received by the Secretaries by September 6 will appear in the September issue of this BULLETIN. By invitation of the Program Committee, Professor J. L. Walsh will deliver an address on *Interpolation by rational functions*.

LEXINGTON, KENTUCKY, November 29-30, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, note later than November 1. All these will appear in the November issue of this BULLETIN. By invitation of the Program Committee, addresses will be delivered by Professor L. L. Dines on *Convex domains and linear inequalities*, and by Professor C. G. Latimer on *The arithmetic of generalized quaternions*.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 30, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than November 2. All these will appear in the November issue of this BULLETIN.

ST. LOUIS, MISSOURI, ANNUAL MEETING, December 31, 1935-January 3, 1936.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 26, 1935. Abstracts received by the secretaries by November 2 will appear in the November issue of this BULLETIN. By invitation of the Program Committee, addresses will be delivered by Professors Szegö and Synge.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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